Consumer Search with Price Sorting*

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Abstract

This paper considers costly sequential search with price sorting in an online-shopping platform in which either ascending or descending price sorting can be applied prior to the search process. A fraction of consumers can search costlessly, while the remaining consumers have identical and positive search costs. The model generates price dispersion in the unique symmetric equilibrium for each type of price sorting, as long as product searches are not perfectly accurate. Moreover, if consumers can choose the type of price sorting, ascending price sorting is chosen more frequently in markets in which product searches are more accurate. Finally, when search costs are small, the availability of price sorting improves consumer surplus but has no impact on industry profit.

Keywords: consumer search, product differentiation, sorting, price dispersion

JEL classifications: D43, D83, L13

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1 Introduction

Price dispersion in homogeneous product markets is a well-documented phenomenon in both offline and online markets (see Baye et al. (2006) for a survey). Ever since Stigler introduced the term "search" in his seminal article "The Economics of Information" (Stigler 1961), many researchers have been trying to rationalize price dispersion using search-theoretical models, in which consumers incur a positive cost of acquiring each additional price quote. Examples include Reinganum (1979), MacMinn (1980), Burdett and Judd (1983), Rob (1985), Stahl (1989, 1996), and Janssen and Moraga-González (2004). The idea behind these models is that firms charge different prices because it is costly for consumers to identify the lowest price due to imperfect information.

The rapid technological growth of the Internet and the flourishing electronic commerce have been enabling consumers to get better informed about prices using various price-searching tools. One of the most commonly observed tools in online-shopping platforms is price sorting. For example, both Amazon.com and eBay.com give customers the option to sort products by price either from low to high, or from high to low. If a consumer can choose how to sort prices prior to the search process, with homogeneous products, there is no doubt that the consumer would choose to sort by price from low to high and therefore costlessly identify the product with the lowest price. Thus, with price sorting, all the previously mentioned search models yield the Bertrand result: firms sell products at marginal cost, and price dispersion disappears among firms that have positive market shares. Yet, in online-shopping platforms, there is price dispersion despite consumers' ability to price sort.

Why, then, do we still observe price dispersion in online markets where consumers can price sort by "low price first"? Further, why do consumers have the option of sorting by "high price first" when they could instead use ascending price sorting? This paper proposes an answer to both of these questions by recognizing that in general, product searches in online-shopping platforms are difficult to target perfectly, and the cheapest results are often not at all what one is looking for. For example, consider a consumer who searches for "Garmin Forerunner 610" under the category of "Electronics" at Amazon.com. After sorting the results by "low price first", the buyer will see a variety of accessories such as USB cables, carrying cases and other irrelevant products for the first five pages, and most of these results have prices less than \$10. However, the buyer can find the right product "Garmin Forerunner 610" immediately if instead, he sorts the results by "high price first".

In this paper, we consider a homogeneous-product market in which product searches are imperfect and consumers have the option of choosing how to sort prices. We assume that in addition to the homogeneous products that consumers are looking for, the search results also include some cheap irrelevant products. Consumers attach zero value to the irrelevant products and will only purchase the relevant products. A market is said to have a higher target accuracy if the fraction of irrelavant products is lower.

Three types of price sorting are considered in this paper: random price sorting, ascending price sorting and descending price sorting. Under random price sorting, searchers sequentially sample the products in a random order, as is commonly assumed in the literature. We use random price sorting as the underlying model to study the impacts of the other two types of sorting. When ascending (or descending, respectively) price sorting is applied, products are displayed from low prices to high prices (or from high prices to low prices, respectively). Consumers sample the products sequentially in the same order as they are displayed. Following Stahl (1989), we assume that a fraction of consumers can search costlessly, while the remaining consumers have identical and positive search costs.

We first assume that the type of price sorting (random, ascending or descending) is publicly known and taken as given. For each scenario, we find that there is a unique symmetric equilibrium,[1] in which firms take the same mixed pricing strategy. Thus, our model generates price dispersion. Intuitively, the existence of shoppers rules out the possibility that firms' equilibrium price distribution having atoms at prices above the marginal cost. On the other hand, each firm is able to earn a positive expected profit because there is positive probability that this firm becomes a monopoly in the market, i.e., the only firm that offers the relevant product. This implies that the equilibrium price distribution cannot be degenerate at the marginal cost. Hence, there are no pure-strategy symmetric equilibria.

In online-shopping platforms, searching usually means clicking the links in the website. The cost of each search is believed to be quite small. Throughout this paper, we focus on the small-search-cost situations so that in equilibrium, search always takes place whenever irrelevant products are discovered. In particular, under ascending price sorting, consumers never stop searching on until all irrelevant products have been sampled, so that they end up purchasing the lowest-priced relevant product due to the ascending order of prices. Under random and descending price sortings, consumers stop searching and make a purchase only when the observed product is relevant and with a price below some reservation price.

We then study how total welfare, consumer surplus and industry profit change with the type of price sorting. We find that with small search costs, both ascending and descending price sortings have no impact on industry profit. That is, the equilibrium expected profit of a high-type firm is the same under each type of price sorting. This is because under each sorting regime, the upper bound of the support of the equilibrium price distribution should be equal to the monopoly price. Note that charging the monopoly price gives a high-type firm the same expected profit, regardless of the type of sorting, because consumers will buy from this firm if and only if all other firms are offering irrelevant products. Since any price in the equilibrium price support gives the same expected profit, a high-type firm should earn identical expected profits under all sorting patterns.

[1] Following Stahl (1989), this paper only focuses on symmetric NE. In an asymmetric equilibrium, consumers have different posterior distributions on prices based on their observations, which makes it difficult to characterize the search behavior and firms' pricing strategies.

We also find that, compared to random price sorting, both ascending and descending price sorting improve total welfare. Intuitively, when the search cost is small, in equilibrium, consumers always end up purchasing the relevant product as long as there are at least one high-type firms in the market. Thus, price sorting affects total welfare only through the total costs of search activities. In other words, the less frequently consumers search, the higher total welfare will be. Note that, both ascending and descending sorting patterns provide useful price information that prevents consumers from inefficient searches. More precisely, with small search costs, consumers under random price sorting almost always want to continue searching because the benefit of making an additional sample is very likely to be higher than the search cost. However, under ascending price sorting, a consumer who have sampled a relevant product will stop searching, because the next product is still a relevant one but with a higher price. Similarly, a consumer under descending price sorting never searches on when observing an irrelevant product. Thus, consumers search less frequently under both ascending and descending price sortings.

It is thus an immediate consequence that consumer surplus will increase when price sorting is applied. Since consumer surplus is affected by both equilibrium prices and total search costs, this implies an interesting fact that the main impact of price sorting is to reduce the total occurrence of search costs, and its influences on pricing strategies are negligible. Actually, as the search cost approaches zero, sorting only yields second-order effects on firms' equilibrium pricing strategies. In other words, in case of small search costs, the price effect of sorting is dominated by the search cost effect.

Finally, instead of fixing the sorting regime exogenously, we let consumers choose how to sort products by price. We find that, with small search costs, random price sorting is never chosen in equilibrium. In other words, consumers always take advantage of sorting options whenever they are available. This is because switching from random price sorting to either ascending or descending price sorting benefits consumers by saving their total search costs, with only negligible influence on their purchase surplus. Morever, consumers choose ascending price sorting (or descending price sorting, respectively) if the market has high (or low, respectively) target accuracy. This is because, given the optimal search behaviors under each sorting pattern and the negligible price effects due to small search costs, ascending price sorting can better save the total search costs in a market in which product searches are more accurate.

There has been a rich literature on search. Stigler (1961) first thought of the "search" phenomenon as an economically important problem, and characterized the optimal search behavior when a searcher can choose a fixed sample size, from which he takes the best alternative. McCall (1970) considered sequential search and used the theory of optimal stopping rules to study unemployment in labor market. Both Stigler (1961) and McCall (1970) assumed that the distribution of alternatives is exogenously given and constant. Our model differs from theirs in that we have considered both sides of the market and derived an equilibrium of price dispersion with optimizing consumers and firms.

As we mentioned before, our model is related to the branch of the search literature concerned with price dispersion. Rob (1985) studied a model of sequential search with heterogeneous consumer search costs in a Nash-Stackelberg game. He assumed that each firm acts as a Stackelberg leader so that consumers are able to observe deviations by firms before they actually search. Stahl (1989, 1996) considered the same situation, but with the Nash paradigm, which assumes that consumers know only the equilibrium price distribution and observe the actual price only when a search is made. Stahl (1989) assumed that there are two types of consumers: shoppers, who have zero search costs, and searchers, who have a common positive search cost. Stahl (1996) considered a more general distribution of consumers' search costs, which is atomless except possibly for a mass of shoppers. All the above models yield equilibria in which firms take mixed pricing strategies, thus produce price dispersion. Reinganum (1979) considered a sequential search model in which firms have different production costs. In equilibrium, each firm takes pure pricing strategy based on its marginal cost, and price dispersion is thus a result of cost heterogeneities. Finally, price dispersion can be generated by other models with fixed sample size search, including MacMinn (1980), Burdett and Judd (1983), and Janssen and Moraga-González (2004). These models, however, fail to provide explanations for price dispersion when price sorting is available.

There are other search models in the literature which do not generate price dispersion. Diamond (1971) considered sequential searches through homogenous goods and found that as long as the search cost is positive, the only equilibrium is that all firms set the monopoly price, which is known as the Diamond Paradox.[2] Wolinsky (1986) resolved the Diamond Paradox by introducing horizontal product differentiation. Anderson and Renault (1999) reconsidered Wolinsky (1986)'s model by introducing the heterogeneity of consumer tastes and the degree of product differentiation. Both Wolinsky (1986) and Anderson and Renault (1999) derived a symmetric equilibrium in which search takes place and firms charge the same price. Price sorting has no impact on the above models because there is only one price in equilibrium.

Our work is also related to the strand of literature which considers non-random sequential search. Weitzman (1979) considered a situation in which several heterogeneous alternatives are available for search, and the optimal search policy should specify not only when to terminate search, but also in which order the searcher should search on. Our model differs from his in two ways. First, Weitzman (1979) allowed the searcher to choose the best search order. In our model, however, the order that consumers sample the products is fully determined by firms' prices and the type of price sorting. Second, while Weitzman (1979) only studied the optimal search policy, our model explores the whole market equilibrium, including firms' pricing strategies, which endogenously affect consumers' search order.

Some recent non-random search models include Arbatskaya (2007), Armstrong et al (2009) and Zhou (2011), who assumed that the order in which consumers search the products is exogenously

[2] In Diamond's model, the Bertrand result is obtained when the search cost equals zero. Thus, as the search cost approaches zero, the equilibrium price changes discontinuously from the monopoly level to the competitive level.

fixed and common knowledge. Arbatskaya (2007) considered a market with homogenous goods, where consumers search only for better prices. The model generates an equilibrium in which firms' prices decline in the order of search. Armstrong et al (2009) and Zhou (2011), on the other hand, considered horizontal product differentiation so that consumers search both for price and product fitness.[3] In contrast to Arbatskaya (2007), both Armstrong et al (2009) and Zhou (2011) found equilibria in which prices increase in the order of search. The reason is that later-searched firms have more monopoly power than earlier-sampled one. Our model differes from the above ones in that the actual search order in our model can be affected by both consumers' and firms' behaviors. For example, if customers sort by price from low to high, then a firm can make its product being sampled earlier by charging a lower price.

The rest of this paper is organized as follows. Section 2 describes the market and three types of price sorting are introduced to the search model. Section 3 considers the case in which the type of price sorting is fixed and publicly known, and derives the equilibria for the three types of sorting, respectively. Section 4 studies the impacts of ascending/descending price sorting, using the case of random price sorting as the underlying model. Section 5 reconsiders the game by assuming that the type of price sorting is chosen by consumers instead of exogenously given. Section 6 extends our model to a more realistic case: more than one products are displayed in each web page, and shows that our main results still hold. Section 7 concludes and discusses possible extensions in the future. All the technical proofs are included in the Appendix.

2 The Model

Consider an oligopoly setting in which N firms compete in selling a homogeneous product to a large group of consumers. Each firm has one unit of the product for sale, with identical production cost normalized to zero. It is common knowledge that a firm's product is either relevant, with probability r, or irrelevant, with probability $1-r$, with $0<r<1$. Firms privately observe their own types of the products. For ease of notation, a relevant product is called a high-quality product, and all the irrelevant products are called low-quality products. Similarly, firms offering relevant (or irrelevant, respectively) products are called high-type (or low-type, respectively) firms. Thus, the probability r characterizes the target accuracy of the market. The higher r is, the more accurately consumers can target their desirable products through searches. While consumers attach zero value to irrelevant products, each relevant product gives the same utility for all consumers, which is given by $a>0$.

On the demand side is a mass of consumers, the size of which is normalized to one. Each consumer wishes to purchase at most one unit of the product. The consumer's surplus of purchase

[3] Armstrong et al (2009) allowed one firm to be prominent, which will be sampled first by all consumers. Other firms will be sampled in a random order once the prominent firm's offer is rejected. Zhou (2011) generalized Armstrong et al (2009) by studying the case where the order in which firms are sampled is completely given.

is given by,

$$u = q - p,$$

where $q \in \{a, 0\}$ is the product quality, and p is the price charged by the firm. Throughout this paper, we only study price strategies for high-type firms—firms that offer relevant products. Without loss of generality, we assume that low-type firms always charge zero price and receive zero profit. Consumers have the option of quitting the market without making any purchase, in which case, both the firm and the consumer receive a utility of zero.

There are two types of consumers. A fixed proportion $\mu \in (0, 1)$ of consumers are assumed to be "shoppers", who costlessly observe all firms' prices and qualities. The remaining proportion $1 - \mu$ of consumers are "searchers", who initially have imperfect information about the prices and the qualities of firms' products. However, searchers can sequentially sample the products to find out these information.

As is commonly assumed in the literature, the first sample is free, but each subsequent sample costs the searcher $c > 0$. After incurring the search cost c, the searcher perfectly learns both the price and the quality of the next product. The searcher then decides whether to continue his search. The search process ends either when the searcher decides to stop sampling, or when all of the N products have been sampled. Finally, the consumer's search is with perfect recall so that the searcher can purchase from the previously sampled firm that gives the highest surplus at no additional search cost.

Prior to the search process, searchers may sort the products by their prices. Three sorting regimes are considered: random price sorting (R), ascending price sorting (A) and descending price sorting (D). Under random price sorting, searchers sample the products in a random order, as is commonly assumed in the literature. When ascending (or descending, respectively) price sorting is applied, the products are displayed, and thus sampled from low prices to high prices (or from high prices to low prices, respectively). Thus, different from previous literature, our paper studies the situation where firms' actions can affect consumers' search order. Finally, we assume that sorting only gives the searcher the information that one price is higher or lower than the other. Searchers do not observe the exact prices unless samples are made.

We will first study the case in which the type of price sorting is exogenously given and publicly known to all parties. Our objective is to derive the firms' optimal pricing strategies, as well as searchers' optimal search policy, under each type of sorting. Later we will consider the case where the price sorting is endogenously determined by searchers. Specifically, choosing the price sorting becomes part of searchers' strategy. And firms set prices taking into account the fact that searchers always choose the price sorting that gives the highest expected purchase surplus.

The timing of the game is as follows. At the beginning, nature draws each firm's quality type. All parties publicly learn the type of price sorting. Firms only observe their own qualities, and simultaneously set their prices according to their quality types and the sorting option. Trades take place in the next stage. The shoppers observe all the prices and qualities, and purchase the product that gives the highest surplus, provided that this surplus is non-negative; otherwise they leave the market without making any purchases. Searchers, on the other hand, search optimally and decide which product to buy as long as the purchase surplus is non-negative.

3 The Equilibrium When Price Sorting is Fixed

The equilibrium concept we use in this paper is perfect Bayesian equilibrium. We focus on a symmetric equilibrium in which high-type firms are taking the same pricing strategy.

Given the type of price sorting $S \in \{R, A, D\}$, high-type firms' pricing strategy can be denoted by a cumulative distribution function $F_S(p)$, with $\overline{p}_S$ and $\underline{p}_S$ being the mimimum and maximum element in its support,[4] which satisfy that $0 \leq \underline{p}_S \leq \overline{p}_S \leq a$.

The market equilibrium under price sorting $S \in \{R, A, D\}$ thus consists of high-type firms' price strategy $F_S(p)$ and a search policy, such that (i) given firms' price distributions, the search policy is optimal for searchers; and (ii) each high-type firm finds it optimal to price according to the distribution $F_S(p)$, given that consumers follow the search policy, and that all other high-type firms follow the strategy $F_S(p)$.

3.1 Optimal Consumer Search

Given firms' pricing strategy $F_S(p)$, a searcher would search on if and only if the expected benefit of doing so exceeds his total subsequent search costs. Searches are more likely to take place when the search cost c per sample is small. Throughout this paper, we will only study the situations in which search always occurs regardless of the type of price sorting. More precisely, under ascending price sorting, a searcher should search on whenever observing a low-quality product. For the cases of random and descending price sorting, we assume that the search cost is sufficiently small so that searches can take place even when a high-quality product is observed. This says that, while low-quality products are never acceptable, a high-quality product does not necessarily stop searchers from searching on. In particular, searchers stop sampling only when the observed high-quality product's price is low enough.

The reason that we focus on the small-search-cost situations is twofold. First, an additional sample gives consumers two possible types of benefits, either when the next product has a better quality, or when it has a lower price. In addition to guaranteeing that search always takes place under all price sortings, small search costs allow us to examine both types of benefits.[5] Second,

[4] There may be gaps in the price support. For example, the support can be $[\underline{p}_S, \alpha] \cup [\beta, \overline{p}_S]$, where α and β are such that $\underline{p}_S < \alpha < \beta < \overline{p}_S$.

[5] While both types of benefits can exist under random price sorting, only the first (or second) type exists under ascending (or descending) price sorting.

our real world examples are online purchases of goods and services. A search in this case always means clicking a link in the website. Thus, the cost of making each sample is believed to be quite small.

The optimal search policy is quite simple under ascending price sorting. Consumers never search on when observing a high-quality product since the next sample is also of high quality but with a higher price. Thus, the optimal search policy is to keep sampling as long as the observed product is of low quality, and buy immediately from the first high-type firm that has been sampled.

Under random price sorting, for a low search cost, searchers always continue their searches when observing low-quality products. Now suppose $p \in [\underline{p}_R, \overline{p}_R]$ is the lowest price of all the high-quality products that have been sampled. Then the expected benefit of sampling one more product is given by

$$\phi_R(p) = r \int_{\underline{p}_R}^{p} (p - x) dF_R(x). \tag{1}$$

Since $\phi_R(p)$ is increasing in p, there exists a unique solution $\hat{p}_R \in [\underline{p}_R, \overline{p}_R]$ that solves $\phi_R(p) = c$, such that $\phi_R(p) \geq c$ if and only if $p \geq \hat{p}_R$. It follows from Meir Kohn and Steven Shavell (1974), Wolinsky (1986), and Stahl (1989) that, the optimal stopping rule under random sorting is to sample the next product if and only if the lowest observed price p is higher than the reservation price $\hat{p}_R$.[6] One can note that this optimal stopping rule is "myopic" in the sense that when deciding whether to search on, consumers always behave as if the next product was the only one left to search.

Finally, let us consider the optimal search policy for descending price sorting. Since the low-type firms always charge price zero, a searcher would never search on when oberving a low-quality product because the next sample is still a low-type. The searcher would buy from the high-type firm that is sampled last. Now we study whether a searcher should search on when all the sampled products so far are of high-quality.

A searcher is said to be at stage n of the search process if he has sampled n high-quality products (with $N - n$ product left unsampled), for $n = 1, ..., N - 1$. Now consider a searcher at stage n, with $p \in [\underline{p}_D, \overline{p}_D]$ as the lowest price he observed. According to the decreasing order of prices, p should be the price of the last sampled product. According to Bayes' Rule, the expected benefit of sampling an additional product is given by

$$\phi_D(p, n) = \int_{\underline{p}_D}^{p} (p - x) d \left(\frac{1 - r + rF_D(x)}{1 - r + rF_D(p)} \right)^{N-n}, \text{ for } n = 1, ..., N - 1. \tag{2}$$

Unlike the case of random price sorting, the expected benefit of sampling the next product under descending price sorting depends not only on the current price, but also on the number of

[6] It does not matter how we specify the stopping rule when $p = \hat{p}_R$. In the next section we will show that the optimal price distribution has no atoms, so that the event $p = \hat{p}_R$ has zero probability in equilibrium.

products left unsampled. To understand expression (2), let random variable $\hat{X}$ be the price of the next product, where $\hat{X}$ equals zero if the next product is of low-quality, and $\hat{X} \in [\underline{p}_D, p]$ if the next product is a high-type. Given the current price p, and that there are $N-n$ products left unsampled, the cumulative distribution function, $G_D(x; p, n)$, associated with variable $\hat{X}$ is given by

$$\begin{aligned} G_D(x;p,n) &= \Pr\{\hat{X} \leq x | p, n\} \\ &= \left(\frac{1-r+rF_D(x)}{1-r+rF_D(p)} \right)^{N-n}, \text{ for any } x \in [\underline{p}_D, p], \end{aligned}$$

since $\hat{X} \leq x$ implies that all the remaining $N-n$ products have prices lower than x.

Thus, the expected benefit of observing $\hat{X}$ is given by

$$\phi_D(p,n) = \int_{\underline{p}_D}^{p} (p-x) dG_D(x;p,n),$$

which gives expression (2).

One can prove that $\phi_D(p,n)$ is increasing in p as long as $(1-r+rF_D(p))^{N-n} - (1-r)^{N-n}$ is log-concave in p.[7] Define $\hat{p}_{D,n}$ as the reservation price at stage n, which solves $\phi_D(p,n) = c$. Thus, at stage n, the benefit of sampling the next product exceeds the search cost if and only if the current price p is higher than the reservation price $\hat{p}_{D,n}$ at stage n.

The following lemma states that with a group of increasing reservation prices, the optimal stopping rule under descending price sorting is "myopic" and fully characterized by the reservation prices.

Lemma 1 *Suppose $\hat{p}_{D,1} \leq \hat{p}_{D,2} \leq ... \leq \hat{p}_{D,N-1}$, then the optimal search policy under descending price sorting is that, at stage n, a searcher continues sampling if and only if his current price is higher than $\hat{p}_{D,n}$, for any $n = 1, ..., N-1$. In other words, the searcher stops sampling and purchases from the current firm at the first stage at which the observed price falls below the corresponding reservation price.*

The intuition is as follows. At each stage, a searcher would obviously keep sampling if his current price is higher than the reservation price. However, when the current price is below the reservation price, the expected benefit of making one more sample at that stage is lower than the search cost. Moreover, since the reservation prices are increasing, the expected benefits of searching on at future stages will never cover the subsequent search costs. Hence, the total benefits of searching on is

[7] We will show later that log-concavity is actually satisfied for the optimal price distribution under descending sorting.

strictly less than the total search costs, even when future behaviors are taken into consideration. As a result, searchers will stop searching and buy at the current price. Note that the assumption of increasing reservation prices is crutial in obtaining a "myopic" stopping rule. To be precise, it makes the next product less worth sampling as fewer products are left unsampled, so that if searchers find that an additional sample is not worth it in the current stage, it is never worth it in the future. The next lemma says that the reservation prices are indeed increasing if the total number of firms N is not large.

Lemma 2 *There exists a positive number $\overline{N}$ such that $\frac{\partial \phi_D(p,n)}{\partial n} < 0$ as long as $N < \overline{N}$.*

Recall that the reservation price $\hat{p}_{D,n}$ solves $\phi_D(p,n) = c$. Thus, the condition $\frac{\partial \phi_D(p,n)}{\partial n} < 0$ immediately implies that $\hat{p}_{D,n}$ is increasing in n. Note that $\frac{\partial \phi_D(p,n)}{\partial n} < 0$ is just a restatement that searching on becomes less beneficial along the search process. The above lemma states that this is true when N is small. Intuitively, under descending price sorting, low-quality products only appear at the back of the sequence. In a market with a small number of products, the fewer products remaining unsampled, the more likely that searchers will find the next product to be a low-quality, which results in a lower expected benefit.

However, $\frac{\partial \phi_D(p,n)}{\partial n} < 0$ may not always hold if there are a large number of products. In this case, searchers are "safe" at the beginning of the search process in the sense that the next product is not likely to be a low-quality. Then searching on can become more beneficial along the search process because with fewer products left, the price of the next high-quality product will be lower under descending price sorting, which brings a higher expected benefit to searchers. The optimal stopping rule for this case can be complex and no longer have the "myopic" property. In particular, searchers may decide to search on even when the benefit of sampling the next product in the current stage is smaller than the search cost, as long as the net benefits of future searches are high. Throughout this paper, we simply assume that the number of products in the market is not large, so that $\frac{\partial \phi_D(p,n)}{\partial n} < 0$ for any $n = 1, ..., N-1$.

3.2 Optimal Pricing Strategy

In this section, we will derive the high-type firms' optimal pricing strategies for each type of price sorting. For the case of random price sorting, we use a method similar to Stahl (1989)'s. A consumer reservation price $\hat{p}_R$ is exogenously fixed, conditional on which we derive the optimal price distribution for high-type firms. The obtained price distribution will again result in a new reservation price $\hat{p}'_R$, according to the previous section. Finally, the equilibrium requires that the pre-given reservation price is consistent, that is, $\hat{p}_R = \hat{p}'_R$. This condition gives the equilibrium reservation price.

Similarly, for the case of descending price sorting, we exogenously fix a group of reservation prices: $\hat{p}_{D,1} \leq \hat{p}_{D,2} \leq ... \leq \hat{p}_{D,N-1}$, and solve for high-type firms' optimal price distribution.

Finally, the pre-given reservation prices should be equal to those derived from the optimal price distribution, which gives the equilibrium.

3.2.1 Random Price Sorting

This subsection considers the case of random price sorting. Let $\hat{p}_R \in (0, a)$ be a pre-given reservation price.[8] Then, the optimal search rule can be stated as: continue to sample if the observed product is a low-type, or if the observed product is a high-type but with a price higher than $\hat{p}_R$; stop sampling and purchase the first high-quality product as long as its price is lower than $\hat{p}_R$. Let $F_R(p; \hat{p}_R)$ be the high-type firms' optimal price distribution conditional on the above search policy. Our first lemma in this section shows that this price distribution has no atoms.

Lemma 3 *$F_R(p; \hat{p}_R)$ is atomless.*

The intuition for Lemma 3 is that under random price sorting, a high-type firms can discretely increase its demand by slightly lowering its price below the atom. Lowering the price obviously increases shoppers' demand discretely. Moreover, a lower price never hurts the demand from searchers. This is because when the price decreases, while the order that the firm is sampled remains unchanged, the purchase surplus becomes higher so that the firm is more likely to attract the returning searchers who have sampled all the products.

Recall that $\overline{p}_R$ and $\underline{p}_R$ are the maximal and minimal elements of the price support. The following lemma gives all the possible values of $\overline{p}_R$.

Lemma 4 *The upper bound of the price support, $\overline{p}_R$, equals either $\hat{p}_R$ or a.*

If $\overline{p}_R < \hat{p}_R$, then only searchers will buy from the firm that charges the highest price $\overline{p}_R$, which happens when this firm is first sampled by some searcher.[9] However, this firm can do strictly better by increasing its price from $\overline{p}_R$ to $\hat{p}_R$, contradicting the optimality of the price distribution. Similarly, if $\hat{p}_R < \overline{p}_R < a$, then charging price a is strictly better than charging $\overline{p}_R$. Thus, the upper bound of the price support should be either $\hat{p}_R$ or a.

Since we focus on the small-search-cost case in which searchers keep sampling when the observed high-quality product has a high price, the reservation price should be strictly lower than the upper

[8] When $\hat{p}_R \geq a$, searchers almost never search on when they observe high-quality products. Since we assume that searches can take place when a high-quality product is observed, we must have $\hat{p}_R$ strictly lower than a.

[9] Shoppers will buy at price $\overline{p}_R$ only when this firm is the unique high-type firm in the market, that is, when a monopoly exists.

bound the price support. Thus, according to lemma 4, the only case we will consider is when $\overline{p}_R = a$.

Note that given the price distribution $F_R(p;\hat{p}_R)$, the demand of a high-type firm, called firm i, is

$$D(p) = \begin{cases} \mu \left[1 - rF_R(p;\hat{p}_R)\right]^{N-1} + \frac{(1-\mu)\left[1-(1-r\hat{F}_R)^N\right]}{rN\hat{F}_R} & \text{if } \underline{p}_R \leq p < \hat{p}_R \\ \left[1 - rF_R(p;\hat{p}_R)\right]^{N-1} & \text{if } \hat{p}_R < p < a \end{cases}, \tag{3}$$

where $\hat{F}_R = F_R(\hat{p}_R;\hat{p}_R)$ is the probability that a high-type firm's price is below the reservation price.

To understand the demand function, note that by charging a price above $\hat{p}_R$, firm i cannot attract searchers when they sample its product at the first time. Thus, both shoppers and searchers buy from firm i if and only if it yields a higher surplus than any of its rivals, which happens with probability $\left[1 - rF_R(p;\hat{p}_R)\right]^{N-1}$.

On the other hand, if $\underline{p}_R \leq p < \hat{p}_R$, then firm i with price p can immediately keep a searcher who just sampled its product. The first expression of the demand function represents firm i's demand from the shoppers. The second term can be rewritten as $\frac{1-\mu}{N}\sum_{m=0}^{N-1}\left(1 - r\hat{F}_R\right)^m$, of which a typical component, $\frac{1-\mu}{N}(1-r\hat{F}_R)^m$, denotes the the fraction of searchers who buy from firm i before having sampled m firms. Note that the probability that a firm is sampled without attracting the searcher immediately is $1 - r\hat{F}_R$.

One can notice from (3) that a high-type firm's demand drops discretely as its price p becomes higher than $\hat{p}_R$. Intuitively, as firm i increases its price slightly above $\hat{p}_R$, it loses the whole "fresh demand"[10] from searchers all of a sudden. Since its demand from shoppers changes continuously, the total demand drops discretely. The discontinuity of the demand function results in a gap in the support of $F_R(p;\hat{p}_R)$, which starts at the reservation price $\hat{p}_R$. Since the demand discretely drops only once, there is at most one gap, because otherwise a high-type firm can always raise its price into the gap without affecting its demand. Thus, the support of $F_R(p;\hat{p}_R)$ is $[\underline{p}_R, \hat{p}_R) \cup [p'_R, a)$ for some $p'_R \in (\hat{p}_R, a)$.

Now we will derive the conditional optimal price distribution. The optimality of $F_R(p;\hat{p}_R)$ requires that a high-type firm earns the same expected profits, denoted by π_R, when charging any price in the support. Moreover, the expected profit by charging any price outside the price support is no greater than π_R. According to the demand function, we have

$$\pi_R = aD(a) = a(1-r)^{N-1}. \tag{4}$$

[10] The notations "fresh demand" and "returning demand" were first used in Zhou (2011). The "fresh demand" represents the demand from searchers who buy immediately after sampling the product. In contrast, the "returning demand" is the demand from searchers who have sampled all the products.

Solving $\pi_R = pD(p)$, our results are summarized in the following proposition.

Proposition 1 *The optimal price distribution conditional on the reservation price $\hat{p}_R \in (0, a)$ is given by*

$$F_R^*(p; \hat{p}_R) = \begin{cases} \frac{1}{r}\left\{1 - \left(\frac{\pi_R}{\mu p} - \frac{(1-\mu)\left[1-(1-r\hat{F}_R)^N\right]}{\mu r N \hat{F}_R}\right)^{\frac{1}{N-1}}\right\} & \text{if } \underline{p}_R \leq p < \hat{p}_R \\ \frac{1}{r}\left[1 - \left(\frac{\pi_R}{p}\right)^{\frac{1}{N-1}}\right] & \text{if } p'_R \leq p < a \end{cases}, \tag{5}$$

where $\underline{p}_R$, $\hat{F}_R$ and p'_R satisfy the following equations

$$\underline{p}_R = \frac{\pi_R}{\mu + \frac{(1-\mu)\left[1-(1-r\hat{F}_R)^N\right]}{rN\hat{F}_R}}, \tag{6}$$

$$\hat{p}_R = \frac{\pi_R}{\mu(1 - r\hat{F}_R)^{N-1} + \frac{(1-\mu)\left[1-(1-r\hat{F}_R)^N\right]}{rN\hat{F}_R}}, \tag{7}$$

$$p'_R = \frac{\pi_R}{(1 - r\hat{F}_R)^{N-1}}. \tag{8}$$

One can easily check that any price outside the support $[\underline{p}_R, \hat{p}_R) \cup [p'_R, a)$ gives a lower profits than π_R. Our last step is to find a consistent reservation price. According to Equation (1) and the condition that $\phi_R(\hat{p}_R) = c$, a consistent reservation price $\hat{p}_R$ should satisfy that

$$r \int_{\underline{p}_R}^{\hat{p}_R} (\hat{p}_R - x) dF_R^*(x; \hat{p}_R) = c. \tag{9}$$

The following proposition gives the existence and uniqueness of the consistent reservation price.

Proposition 2 *When search cost c is sufficiently small, there is a unique $\hat{p}_R$ that solves Equation (9). Moreover, the consistent reservation price increases as the search cost becomes larger.*

As an extreme case, when search cost c approaches zero, the probability $\hat{F}_R$ converges to zero. According to (6)-(8), this means that all $\underline{p}_R$, $\hat{p}_R$ and p'_R converge to the same value $\pi_R = a(1-r)^{N-1}$. Thus, the gap in the price support disappears and the optimal price distribution becomes

$$F_R^*(p) = \frac{1}{r}[1 - \left(\frac{\pi_R}{p}\right)^{\frac{1}{N-1}}], \text{ for } p \in [\pi_R, a).$$

Intuitively, this price distribution is the one as if all consumers are shoppers, that is, all consumers purchase the product with the highest surplus.

Finally, to relate our result to the search literature, one can notice that, in our model, the price dispersion is a result of the assumption that the fraction of shoppers μ is strictly between zero and one. In other words, both shoppers and searchers exist in the market. The equilibrium of our model would be totally different if there were only shoppers or searchers. For example, when all the consumers are searchers, then as long as the search cost is positive (no matter how small it is), all high-type firms charging price a constitutes an equilibrium, in which searchers always purchase from the first firm they sample. This is the well-known Diamond Paradox. On the other hand, when there are only shoppers, then the only equilibrium is that all firms price at their marginal cost, zero, which is exactly the Bertrand result. Hence, our results are consistent with Stahl (1989)'s even when vertical product differentiation is considered. Moreover, while Wolinsky (1986) and Anderson and Renault (1999) succeeded in resolving the Diamond Paradox by introducing horizontal product differentiation, our model shows that vertical differentiation alone is not enough to "get around" the the Diamond Paradox, unless some consumers are allowed to have zero search cost.

3.2.2 Ascending Price Sorting

This subsection studies the equilibrium under ascending price sorting. Recall that when search cost is small, the optimal search pocily is that searchers continue to search if and only if a low-quality product is observed. Let $F_A(p)$ be the high-type firm's optimal pricing strategy, with $\overline{p}_A$ and $\underline{p}_A$ as the highest and lowest elements of the price support, where $0 \leq \underline{p}_A \leq \overline{p}_A \leq a$.

Lemma 5 *$F_A(p)$ is atomless.*

The intuition here is almost the same as that for lemma 3. Under ascending sorting, a lower price not only lets the firm be sampled earlier, but increases the purchase surplus so that a returning searcher is more likely to make a purchase.

Lemma 6 *$\overline{p}_A = a$ and there is no gap in the price support.*

If the upper bound of the price support is strictly below a, then any price between $\overline{p}_A$ and a should result in the same demand as $\overline{p}_A$ does. This obviously contradicts the optimality of $F_A(p)$. Similarly, if there is a gap in the price support, then the high-type firm can always raise its price into the gap without affecting the demand, hence strictly increasing its profits.

Given the price distribution $F_A(p)$ and the consumer's search rule, both shoppers and searchers will purchase from the high-type firm that charges the lowest price. In other words, searchers behave exactly the same as shoppers. A high-type firm's total demand is thus given by

$$D(p) = [1 - rF_A(p)]^{N-1}. \tag{10}$$

Let π_A be the high-type firm's expected profit under ascending price sorting, then

$$\pi_A = aD(a) = a\,(1-r)^{N-1}\,. \tag{11}$$

The optimal price distribution $F_A^*(p)$ can thus be obtained by solving $\pi_A = pD(p)$.

Proposition 3 *When search cost c is lower than ϕ_A, the optimal price distribution under ascending sorting is*

$$F_A^*(p) = \frac{1}{r}[1-\left(\frac{\pi_A}{p}\right)^{\frac{1}{N-1}}],\ \textit{for}\ p \in [\underline{p}_A, a), \tag{12}$$

where

$$\underline{p}_A = \pi_A = a\,(1-r)^{N-1}\,, \tag{13}$$

$$\phi_A = \min_{1\leq n\leq N-1} \phi_A(n), \tag{14}$$

$$\phi_A(n) = \frac{C(N,n)\,(1-r)^n\,r^{N-n}}{\sum_{i=n}^{N} C(N,i)\,(1-r)^i\,r^{N-i}} \int_{\underline{p}_A}^{a} \left[1-(1-F_A^*(p))^{N-n}\right] dx. \tag{15}$$

Here $C(N,n)$ is the number of combinations of n out of N objects, which is given by

$$C(N,n) = \frac{N!}{(N-n)!n!}.$$

Note that any price below $\underline{p}_A$ cannot be a profitable deviation because it gives the same demand as $\underline{p}_A$ does. This verifies the optimality of $F_A^*(p)$.

Expression (15) gives the expected benefit of sampling an additional product after n low-type firms have been sampled, $n = 1, ..., N-1$. Given the increasing price order, and the fact that the first n products have low quality, let random variable $\hat{X}$ be the price of the next product. Obviously, $\hat{X} = 0$ if the next product is a low-type; and $\hat{X} \in [\underline{p}_A, a]$ if the next product is a high-type. Then for any $x \in [\underline{p}_A, a]$, the probability that $\hat{X} > x$ is given by

$$\frac{C(N,n)\,(1-r)^n\,r^{N-n}\,(1-F_A^*(p))^{N-n}}{\sum_{i=n}^{N} C(N,i)\,(1-r)^i\,r^{N-i}},$$

where the denominator is the probability that at least n products are of low quality, and the numerator is the probability that only m products have low quality, with all the other products being high-types with prices higher than x.

Let $G_A(x;n)$ be the cumulative distribution function associated with $\hat{X}$, given that n low-quality products have been observed. Then

$$G_A(x;n) = 1 - \frac{C(N,n)\,(1-r)^n\,r^{N-n}\,(1-F_A^*(p))^{N-n}}{\sum_{i=n}^{N} C(N,i)\,(1-r)^i\,r^{N-i}}, \text{ for } x \in [\underline{p}_A, a],$$

and by definition

$$\phi_A(n) = \int_{\underline{p}_A}^{a} (a-x)\,dG_A(x;n),$$

which immediately gives (15). Finally, the condition $c < \phi_A$ guarantees that searchers never stop sampling until a high-quality product is observed.

3.2.3 Descending Price Sorting

In this subsection, we derive the equilibrium price distribution for descending price sorting, under the assumption that the search cost is sufficiently small. The analyses for random price sorting show that to avoid the Diamond Paradox, we need to assume the existence of shoppers, that is, assume $\mu > 0$. This seems not to be a sufficient condition when descending sorting is under consideration. The following lemma states that even when both shoppers and searchers exist in the market, the Diamond result can still be obtained as long as μ is small.

Lemma 7 *Under descending price sorting, if* $\mu \leq \frac{1-(1-r)^N - rN(1-r)^{N-1}}{1-(1-r)^N - rN(1-r)^{N-1} + r(N-1)}$*, then there exists an equilibrium in which all high-type firms set a price equal to a, and searchers purchase from the first firm that they sample.*

Intuitively, under descending price sorting, by raising its price, a high-type firm can make itself more likely to be sampled earlier, thus capturing a higher "fresh demand" from searchers,[11] inspite of the fact that doing so can lower the damand from shoppers. Thus, the fractions of shoppers and searchers play an important role in determining firms' total demand. Specifically, when μ is low enough, the amount of searchers is so large that the increased "fresh demand" from searchers always outweighs the demand loss from shoppers. Thus, it is profitable for high-type firms to increase their prices to the highest level, a.

For the rest of this section, we assume that $\mu > \frac{1-(1-r)^N - rN(1-r)^{N-1}}{1-(1-r)^N - rN(1-r)^{N-1} + r(N-1)}$ so that the Diamond result does not hold. We aim at solving an equilibrium with price dispersion.

Let $\hat{p}_D = (\hat{p}_{D,1}, ..., \hat{p}_{D,N-1})$ be a set of pre-given reservation prices such that $0 < \hat{p}_{D,1} \leq \hat{p}_{D,2} \leq ... \leq \hat{p}_{D,N-1} < a$.[12] The conditional optimal price distribution for high-type firms is

[11] A higher price may also cause a loss of the demand from the returning searchers who have sampled all the products. But since the later sampled products always have lower prices, this type of demand is very limited (a searcher will go back to purchase a previously sampled product only if all the later sampled products are low-types).

[12] Under descending price sorting, we assume that the search cost is so small that search takes place at each stage as long as the current price is very high.

denoted by $F_D(p;\hat{p}_D)$. Let $\overline{p}_D$ and $\underline{p}_D$ be the upper and lower bounds of the price support, where $0 \leq \underline{p}_D \leq \overline{p}_D \leq a$.

To focus on the interesting cases, we assume that μ is sufficiently large so that $F_D(p;\hat{p}_D)$ is atomless[13]. The situation here is slightly different from those under random and ascending sortings, where undercutting a price atom will obviously not decrease the firm's demand from searchers. For descending price sorting, charging a price slightly below an atom would reduce a high-type firm's "fresh demand" from searchers because other firms that charge the atom price would be sampled ahead of this firm. Our assumption simply states that if there are sufficiently large amount of shoppers, then the loss of "fresh demand" is always dominated by the discrete increase of the demand from shoppers, thus making this deviation strictly profitable.

The next lemma characterizes the upper bound of the price support.

Lemma 8 *When the search cost is sufficiently small, the upper bound of the price support, $\overline{p}_D$, equals a.*

The intuition for lemma 8 is the same as that for lemma 4. Since search takes place at each stage as long as the observed price is high enough, the upper bound of price support should be above the highest reservation price. Thus, a high-type firm that charges price $\overline{p}_D$ has positive demand only if all other firms are low-types. Suppose $\overline{p}_D$ is strictly lower than a, then charging price a is strictly better than charging $\overline{p}_D$ because increasing the price from $\overline{p}_D$ to a will not affect the high-type firm's demand.

For ease of notation, define

$$\hat{p}_{D,0} \equiv \underline{p}_D \text{ and } \hat{p}_{D,N} \equiv \overline{p}_D = a.$$

We now derive the high-type firm's demand function for each price interval, $p \in (\hat{p}_{D,n}, \hat{p}_{D,n+1}]$, where $n = 0, 1, ..., N-1$. Let $D_0(p)$ and $D_1(p)$ be the demand from shoppers and searchers, respectively. Thus, the total demand is given by

$$D(p) = \mu D_0(p) + (1-\mu)\, D_1(p), \tag{16}$$

where $D_0(p) = [1 - rF_D(p;\hat{p}_D)]^{N-1}$ since shoppers buy at price p if and only if all other products are either low-quality or have a price higher than p.

[13] As μ approaches 1, the optimal price distribution is obviously atomless because all consumers are shoppers.

To derive the demand from searchers, note that

$$\begin{aligned} D_1(p) &= \Pr\{\text{searchers buy at price } p\} \\ &= \sum_{k=0}^{N-1} \Pr\{\text{searchers buy at price } p\text{, with } k \text{ prices higher than } p\} \\ &= \sum_{k=0}^{N-1} D_1(p,k), \end{aligned} \tag{17}$$

where $D_1(p,k) = \Pr\{\text{searchers buy at price } p\text{, and there are } k \text{ prices higher than } p\}$, for $k = 0, 1, ..., N-1$.

We have that for any $p \in (\hat{p}_{D,n}, \hat{p}_{D,n+1}]$,

$$D_1(p,k) = \begin{cases} C(N-1,k)\,(1-r)^{N-1-k}\,[r - rF_D(p;\hat{p}_D)]^k, & \text{if } k < n \\ C(N-1,k)\,(1-r+rF_D(p;\hat{p}_D))^{N-1-k}\,[r - rF_D(p;\hat{p}_D)]^k, & \text{if } k = n \\ C(N-1,k)\,(1-r+rF_D(p;\hat{p}_D))^{N-1-k}\left[r - r\hat{F}_{D,k}\right]^k, & \text{if } k > n \end{cases},$$

where $\hat{F}_{D,k} = F_D(\hat{p}_{D,k};\hat{p}_D)$ and $C(N-1,k)$ is the number of combinations of k out of $N-1$ prices, that is,

$$C(N-1,k) = \frac{(N-1)!}{(N-1-k)!k!}, \text{ for any } k < N-1.$$

To understand the expression of $D_1(p,k)$, note that if there are $k < n$ prices higher than p, due to the fact that $p > \hat{p}_{D,n}$, searchers will not accept price p without further searches, which means that searchers buy at price p only if all the products with prices lower than p are low-quality. Thus, in this case, $D_1(p,k)$ equals the probability that k out of $N-1$ prices are higher than p, and the rest $N-1-k$ prices are zero.[14]

If $k = n$, then price p is the $(n+1)\,th$ highest price of all products. Since $\hat{p}_{D,n} < p \leq \hat{p}_{D,n+1}$, searchers never accept any price higher than p, and buy at p immediately without further searches. Thus, $D_1(p,k)$ equals the probability that k out of $N-1$ prices are higher than p, and the rest $N-1-k$ prices are lower than p.

Finally, if $k > n$, due to the increasing order of reservation prices, price p will be immediately accepted by searchers since $p \leq \hat{p}_{D,n+1} \leq \hat{p}_{D,k}$. Thus, searchers buy at p only if they do not accept any prices higher than p, that is, the previous k prices should be higher than the reservation price $\hat{p}_{D,k}$. In other words, $D_1(p,k)$ is the probability that k out of $N-1$ prices are higher than $\hat{p}_{D,k}$, and the rest $N-1-k$ prices are lower than p.

[14] Zero price means the product is of low quality.

According to (16) and (17), the total demand for $p \in (\hat{p}_{D,n}, \hat{p}_{D,n+1}]$ is given by

$$\begin{aligned} D(p) &= \mu \left[1 - rF_D(p; \hat{p}_D)\right]^{N-1} \\ &+ (1-\mu) \sum_{k=0}^{n-1} C(N-1,k)\,(1-r)^{N-1-k} \left[r - rF_D(p; \hat{p}_D)\right]^k \\ &+ (1-\mu)\, C(N-1,n)\,(1-r+rF_D(p;\hat{p}_D))^{N-1-n} \left[r - rF_D(p;\hat{p}_D)\right]^n \\ &+ (1-\mu) \sum_{k=n+1}^{N-1} C(N-1,k)\,(1-r+rF_D(p;\hat{p}_D))^{N-1-k} \left[r - r\hat{F}_{D,k}\right]^k . \end{aligned} \tag{18}$$

Similar to the case of random price sorting, the above demand function is discontinuous at each reservation price $\hat{p}_{D,n}$. Precisely, for each $n = 1, ..., N-1$, we have that

$$\begin{aligned} & \lim_{p \to \hat{p}_{D,n}-} D(p) - \lim_{p \to \hat{p}_{D,n}+} D(p) \\ = \; & (1-\mu)\, C(N-1, n-1) \left(r - r\hat{F}_{D,n}\right)^{n-1} \left[\left(1 - r + r\hat{F}_{D,n}\right)^{N-n} - (1-r)^{N-n}\right] \\ > \; & 0. \end{aligned}$$

The intuition is as follows. When a high-type firm slightly increases its price above $\hat{p}_{D,n}$, its demand from searchers drops discretely conditional on the case that this firm charges the $n-th$ highest price among all firms, because the high-type firm no longer captures all the "fresh demand" from searchers when it is $n-th$ sampled. As a result, there must be $N-1$ gaps in distribution $F_D(p;\hat{p}_D)$'s support. The price support is the union of N intervals, which has the form $\cup_{n=0}^{N-1}[p'_{D,n}, \hat{p}_{D,n+1})$, where $p'_{D,n} \in (\hat{p}_{D,n}, \hat{p}_{D,n+1})$ for any $n = 1, ..., N-1$ and $p'_{D,0} = \hat{p}_{D,0} = \underline{p}_D$.

It follows from the demand function (18) that $D(a) = (1-r)^{N-1}$. Thus, the high-type firm's expected profit is

$$\pi_D = aD(a) = a(1-r)^{N-1}. \tag{19}$$

The conditional optimal price distribution is then given in the following proposition.

Proposition 4 *Let $F_D^*(p;\hat{p}_D)$ be the optimal atomless price distribution conditional on the set of reservation prices such that $0 < \hat{p}_{D,1} \leq \hat{p}_{D,2} \leq ... \leq \hat{p}_{D,N-1} < a$. Then, for any $n = 0, 1, ..., N-1$ and $p \in [p'_{D,n}, \hat{p}_{D,n+1}]$, $F_D^*(p;\hat{p}_D)$ solves*

$$\frac{\pi_D}{p} = D(p),$$

where $D(p)$ is given in (18), $\underline{p}_D$, $\hat{F}_{D,n}$ and $p'_{D,n}$ for $n \geq 1$ are determined by

$$\frac{\pi_D}{\underline{p}_D} = \mu + (1-\mu) \sum_{k=0}^{N-1} C(N-1,k) \, (1-r)^{N-1-k} \left[r - r\hat{F}_{D,k} \right]^k, \tag{20}$$

$$\hat{F}_{D,n} = F_D^*(\hat{p}_{D,n}; \hat{p}_D) = F_D^*(p'_{D,n}; \hat{p}_D), \tag{21}$$

$$\frac{\pi_D}{p'_{D,n}} = \lim_{p \to p'_{D,n}} D(p). \tag{22}$$

To verify the optimality of $F_D^*(p; \hat{p}_D)$, one can easily check that any price outside the support $\cup_{n=0}^{N-1} [p'_{D,n}, \hat{p}_{D,n+1})$ cannot be a profitable deviation.

Moreover, it follows from (20) that $[1 - r + rF_D^*(p; \hat{p}_D)]^{N-n} - (1-r)^{N-n}$ is log-concave in p within the price support, which verifies that the expected benefit of sampling another product at stage n, $\int_{\underline{p}_D}^{p} (p-x) d \left(\frac{1-r+rF_D^*(x;\hat{p}_D)}{1-r+rF_D^*(p;\hat{p}_D)} \right)^{N-n}$, is increasing in p. A set of consistent reservation prices $\hat{p}_D$ thus requires that

$$\int_{\underline{p}_D}^{\hat{p}_{D,n}} (\hat{p}_{D,n} - x) d \left(\frac{1 - r + rF_D^*(x; \hat{p}_D)}{1 - r + r\hat{F}_{D,n}} \right)^{N-n} = c, \text{ for all } n = 1, ..., N-1. \tag{23}$$

Finally, the existence and uniqueness of the set of consistent reservation prices is given in the following proposition.

Proposition 5 *When search cost c is sufficiently small, there is a unique set of reservation prices $(\hat{p}_{D,1}, ..., \hat{p}_{D,N-1})$ that solves Equation (23).*

Similar to the case of random price sorting, as search cost c approaches zero, all reservation prices $\hat{p}_{D,n}$, as well as the lower bound $\underline{p}_D$, converge to $\pi_D = a(1-r)^{N-1}$. The optimal price distribution then becomes the one as if all consumers are shoppers.

4 The Impacts of Price Sorting

In this section, we use random price sorting as the underlying model and study the effects of ascending price sorting and descending price sorting. We explore how price sortings affect market performance. Given the type of price sorting $S \in \{R, A, D\}$, let TW_S, Π_S and CS_S be total welfare, industry profit and consumer surplus, respectively. Since there are two types of consumers, let CS_S^0 and CS_S^1 be the surpluses for shoppers and searchers, respectively. The total consumer surplus is thus a weighed average of CS_S^0 and CS_S^1, depending on the fraction of shoppers. That is, $CS_S = \mu CS_S^0 + (1-\mu) CS_S^1$. Finally, let N_S be the average number of searches made by consumers.

Our first result describes the impact of price sorting on the search intensity.

Proposition 6 *When search cost c is small, compared to random price sorting, consumers search less often under both ascending and descending price sortings. Specifically, we have*

$$\begin{aligned} N_A &< N_D < N_R, \text{ if } r > 1/2; \\ \text{and } N_D &< N_A < N_R, \text{ if } r < 1/2. \end{aligned}$$

Intuitively, for the extreme case in which search is costless, consumers under random price sorting will sample all the products before making a purchase. However, under ascending price sorting, consumers would stop searching when observing the first high-quality product because they know that this product has the lowest price and thus the highest purchase surplus. Similarly, under descending price sorting, the search process terminates when all the high-quality products have been sampled. Thus, both types of sorting reduce the number of searches by providing useful price information to consumers. Finally, it is obvious that search terminates earlier under ascending (descending) price sorting when firms are more (less) likely to be high-type's, that is, when $r > (<) 1/2$. Although this explanation is provided for the extreme case where search cost c approaches zero, our result still holds for positive search costs, as long as c is small.

The following result shows that price sorting always improves the total welfare.

Proposition 7 *When search cost c is small, (i) both ascending and descending price sortings improve total welfare; (ii) ascending price sorting has a greater improvement on total welfare than descending price sorting if and only if r > 1/2. That is,*

$$\begin{aligned} TW_A &> TW_D > TW_R, \text{ if } r > 1/2; \\ \text{and } TW_D &> TW_A > TW_R, \text{ if } r < 1/2. \end{aligned}$$

Total welfare depends on the expected benefits of the trade between firms and consumers, and the total expected costs of the search activities. Note that regardless of the type of sorting, a consumer always ends up purchasing a high-quality product, as long as there is at least one high-type firm in the market. This implies that the expected benefits of the trade should be $a\left[1-(1-r)^N\right]$, which is the same under all types of price sortings. Thus, the less frequently consumers search, the higher total welfare will be. Proposition 7 thus follows immediately from Proposition 6: price sorting always provides useful price information that prevents consumers from inefficient searches, and ascending (descending) price sorting better saves the total occurence of search costs when there are more high-quality (low-quality) products in the market.

The next result describes the impacts of price sortings on industry profits.

Proposition 8 *When search cost c is small, both ascending and descending sortings have no impact on industry profits. That is*

$$\Pi_A = \Pi_D = \Pi_R$$

To explain this result, note that under each type of price sorting, a high-type firm earns the same expected profit when charging any price in the support of the optimal price distribution. In addition, when charging the monopoly price, a, which equals the high-type product's quality,[15] the high-type firm earns the same expected profit regardless of the type of price sorting, because consumers will buy from this firm if and only if all other firms are low-types. Thus, the expected profit of a high-type firm is the same for all types of price sortings. Since a low-type firm always earns zero profit, industry profits should be the same under each price sorting.

Finally, the result on consumer surplus follows immediately from Proposition 7 and 8.

Proposition 9 *When search cost c is small, (i) both ascending and descending price sortings boost consumer surplus; (ii) ascending price sorting has a larger effect on consumer surplus than descending price sorting if and only if $r > 1/2$. That is,*

$$\begin{aligned} CS_A &> CS_D > CS_R, \text{ if } r > 1/2; \\ \text{and } CS_D &> CS_A > CS_R, \text{ if } r < 1/2. \end{aligned}$$

Note that total consumer surplus is a weighed average of CS_S^0 and CS_S^1. Analyses in the previous section show that as search cost c approaches zero, the optimal price distributions for all types of sorting converge to the same limit. This implies that the limits of CS_S^0 are also the same for all $S \in \{R, A, D\}$. Thus, as long as c is small, the impact of price sorting on shoppers' surplus, CS_S^0, is negligible. However, Proposition 10 states that the impact of price sorting on searchers' surplus, CS_S^1, is significant and the same as that on both total welfare and total consumer surplus.

Proposition 10 *When search cost c is small, we have*

$$\begin{aligned} CS_A^1 &> CS_D^1 > CS_R^1, \text{ if } r > 1/2; \\ \text{and } CS_D^1 &> CS_A^1 > CS_R^1, \text{ if } r < 1/2. \end{aligned}$$

[15] Consider that the price approaches a from below.

While Proposition 9 is immediate due to previous results, Proposition 10 needs a detailed proof (See Appendix). Note that searchers' surplus CS_S^1 has two components: searchers' *purchase surplus*, the expected benefits of the trade between firms and searchers, and searchers' total search costs. The intuition behind our results is that when c is small, the main impact of price sorting is to reduce the total occurrence of search costs. The effects of price sorting on firms' profits and consumers' *purchase surplus* are negligible, because the optimal price distribution under each type of sorting has the same limit, as search cost converges to zero. This also explains why price sorting has similar impacts on total welfare and searchers' surplus. Finally, the different effects of the price sortings on searchers' surplus are characterized by Proposition 6.

5 Endogenizing Price Sorting

Until now we have assumed that the type of price sorting is exogenously given and known to each party in the market. However, in many real-world situations, consumers have the option to choose their best type of sorting. This section studies the endogenization of price sorting. As is commonly observed, we assume that there are three types of price sorting available to consumers: random sorting (R), ascending sorting (A), and descending sorting (D). Again, the case of random sorting can be intepreted as searching without sorting.

The timing of the new game is as follows. Firstly, nature draws each firm's quality type. Firms then privately observe their own qualities, and simultaneously set their prices according to their quality types. Consumers, at the same time, choose the type of price sorting.[16] Shoppers observe all the prices and qualities, and purchase the product that gives the highest surplus, provided that this surplus is non-negative; otherwise they leave the market without making any purchases. Searchers, on the other hand, observe the first free sample (p, q), and then search optimally. We still aim at deriving a symmetric perfect Bayesian equilibrium in which high-type firms take the same pricing strategy, and consumers choose the same type of price sorting.

Since choosing the type of price sorting becomes part of consumers' strategy, the equilibrium of the new game should consist of the high-type firm's price distribution $F(p)$, the price sorting $S \in \{R, A, D\}$, and a search policy such that (i) given the price sorting S and the search policy, the price distribution $F(p)$ is optimal for each high-type firm; (ii) given the price distribution $F(p)$ and the type of price sorting S, the search policy is optimal for searchers; and (iii) given the price distribution $F(p)$, sorting S, as well as its associated optimal search policy, is the best option among the three, in the sense that it gives searchers the highest expected surplus.

Let $F_S(p)$ be the equilibrium price distribution when price sorting S is exogenously fixed, for $S \in \{R, A, D\}$. The above definition implies that, when price sorting is endogenized, there are only

[16]Since all consumers are identical ex ante, it is reasonable to assume that they choose the same type of price sorting. Note that shoppers do not care about the type of sorting because they observe all the information at no cost.

three equilibrium candidates $(F_R(p), R), (F_A(p), A)$ and $(F_D(p), D)$.[17] Moreover, to check whether $(F_S(p), S)$ is an equilibrium, we only need to check whether sorting S gives a higher expected consumer surplus than the other two types of sorting, given the price distribution $F_S(p)$. Our result is given in the following proposition.

Proposition 11 *When price sorting is endogenized, with small search cost c, (i) consumers never choose random price sorting in an equilibrium; (ii) when $r > 1/2$, the unique equilibrium is $(F_A(p), A)$, i.e. ascending sorting is chosen in equilibrium; (iii) when $r < 1/2$, the unique equilibrium is $(F_D(p), D)$, i.e. descending sorting is chosen in equilibrium.*

Proposition 11 (i) states that consumers always take advantage of sorting options whenever they are available. The reason is that for a given price distribution, a searcher can always get better off by switching from random sorting to either ascending sorting or descending sorting. This is because with small search costs, both types of price sorting can reduce the number of searches and lower the total costs of search activities. Although the switch of price sorting will change consumers' purchase surplus, the effect is quite small compared to the savings on the total search costs. This reaffirms our claim that the main impact of price sorting is on searchers' total search costs, instead of consumers' *purchase surplus* and firms' profits.

To understand Proposition 11 (ii) and (iii), note that in equilibrium consumers should choose the type of price sorting that best improves searchers' surplus. Considering its main impact, the chosen price sorting should have a larger effect in saving the total search costs. Thus, Proposition 11 is just a restatement that ascending price sorting better saves the total consumer search costs than descending price sorting does if and only if there are more high-type firms in the market.

6 Extension: Multiple Products in Each Sample

This paper assumes that the full product information, price and whether it is irrelavant, is not known to the consumer unless the product is sampled. In online-shopping platforms, all the search results are displayed in several pages. Each page reveals full product information including price and product details. Thus, sampling the products simply means viewing the web pages according to the sorting results. And taking another sample means clicking the "next page" button on the website. Consumers learn all the product information when they go to the new page. Until now, we have been assuming that each page shows only one product, just for analytical convenience. In reality, however, more than one products are observed in each weg page. For example, Amazon.com displays 24 products per page, with all the prices and product discriptions. People who search for flights at Expedia.com can find 10 different flight deals in each page, which shows the origins, destinations, travel dates and prices. The number of items displayed in each page at eBay.com can

[17] We have omitted the associated search policy in the expression of an equilibrium just for simplicity of notation.

be 25, 50, 100 or 200, which is determined by consumers. In this section, we show that our main results still hold even when multiple products are observed in each sample.

Let us reconsider the previous game by assuming that each sample consists of M products. That is, by incurring a search cost c, searchers learn price and quality information of M products at the same time. All other assumptions remain the same as before. We will derive the symmetric equilibrium in which low-type firms set zero price and high-type firms take the same price strategy $F_S(p;M)$,[18] when each type of price sorting $S \in \{R, A, D\}$ is exogenously given.

The method we use to solve the equilibrium is the same as before. We first show that consumers' optimal search policy are fully characterized by some reservation prices, given that all high-type firms follow the price distribution $F_S(p;M)$. The reservation price can be calculated an functions of the price distribution. Then, we exogenously fix the reservation price(s) and solve the conditional optimal price distribution for high-type firms. Finally, in equilibrium, the pre-given reservation price(s) should be consistent with the obtained optimal price distribution. That is, the optimal reservation price(s) calculated from the obtain price distribution should be exactly equal to the pre-given reservation price(s).

6.1 Random Price Sorting

We first solve the symmetric equilibrium for random price sorting. Given the high-type firms' price distribution $F_R(p;M)$, and the current lowest price, p, of all sampled high-quality products, the expected benefit of sampling the next product is

$$\phi_R(p;M) = \sum_{s=1}^{M} C(M,s) r^s (1-r)^{M-s} \int_{\underline{p}_R}^{p} (p-x) dF_R^{(s)}(x;M), \tag{24}$$

where $F^{(n)}(p)$ is the cumulative distribution function of the lowest of n independent random variables with the same distribution $F(p)$.

To understand the benefit (24), note that with probability $C(M,s)r^s(1-r)^{M-s}$, there are s high-quality products in the next sample. Then the lowest price of the high-quality products has distribution $F_R^{(s)}(x;M)$, given high-type firms' price distribution $F_R(p;M)$. Thus, each typical component of the right-hand side of Expression (24) is the expected incremental utility from getting a lower price in the next sample, conditional on that s of M products in the sample are high-types.

Define the reservation price $\hat{p}_R^M$ as the solution to the equation $\phi_R(p;M) = c$. According to previous analysis, the optimal stopping rule in this case is fully characterized by the reservation price $\hat{p}_R^M$: at any stage of the search process, consumers continue searching if and only if the current lowest price p is above the reservation price $\hat{p}_R^M$.

[18] The previous game is the case in which $M = 1$.

Now we derive high-type firms' optimal price distribution $F_R(p;M,\hat{p}_R^M)$ for any given reservation price $\hat{p}_R^M$. A high-type firm's demand is given as follows.

$$D(p)=\begin{cases} \mu\left[1-rF_R(p;M,\hat{p}_R^M)\right]^{N-1} \\ +(1-\mu)\sum_{t=0}^{\frac{N}{M}-1}\frac{M}{N}\left[1-rF_R(p;M,\hat{p}_R^M)\right]^{M-1}\left(1-r\hat{F}_R^M\right)^{mt} & \text{if } p<\hat{p}_R^M \\ \left[1-rF_R(p;M,\hat{p}_R^M)\right]^{N-1} & \text{if } p>\hat{p}_R^M \end{cases},$$

where $\hat{F}_R^M=F_R(\hat{p}_R^M;M,\hat{p}_R)$ and we treat $\frac{N}{M}$ as an integer for analytical convenience.

When price p is below the reservation price, searchers buy at p if and only if they reject all the previously sampled products and price p is the lowest price in the current sample. Note that $\frac{M}{N}\left[1-rF_R(p;M,\hat{p}_R^M)\right]^{M-1}\left(1-r\hat{F}_R^M\right)^{mt}$ is the probability that price p is the lowest price in the $(t+1)\,th$ sample, and all the first t samples ahead of p are rejected by searchers.

According to the definition of the equilibrium, the optimal price distribution can be solved by the constant profit condition $\pi_R=pD(p)$ for any p within the price support.

Finally, the equilibrium reservation price is solved according to the consistence condition

$$\begin{aligned}\phi_R(\hat{p}_R^M;M) &= \sum_{s=1}^{M}C(M,s)r^s\left(1-r\right)^{M-s}\int_{\underline{p}_R}^{\hat{p}_R^M}(\hat{p}_R^M-x)dF_R^{(s)}(x;M,\hat{p}_R^M) \\ &= c.\end{aligned}$$

6.2 Ascending Price Sorting

The equilibrium under ascending price sorting is simple. As long as the search cost c is small, the optimal search rule under ascending price sorting is to stop searching only when a high-quality product is observed. Thus, searchers behave like shoppers in the sense that they always purchase the product with the highest surplus. Hence, the high-type firms' optimal price distribution becomes the same as that in Proposition 3.

6.3 Descending Price Sorting

Now we solve the equilibrium for descending price sorting. Suppose each high-type firm takes the price strategy $F_D(p;M)$. Consider a state in which k pages of products have been sampled, and the last product in the current page is of high quality with price p, for $k=1,...,\frac{N}{M}-1$. We call this stage k of the search process. Next, we will calculate the expected benefit of sampling an additional page, given the current state (p,k).

Let $\sigma(t;p,k,M)$ be the probability that there are t high-quality products left, conditional on the current state (p,k). Then, for any $t=0,1,...,N-kM$, we can easily get

$$\sigma(t;p,k,M)=\frac{C(N-kM,t)\left[rF_D(p;M)\right]^t\left(1-r\right)^{N-kM-t}}{\left[1-r+rF_D(p;M)\right]^{N-kM}}. \tag{25}$$

Given the current state (p,k), the price of each high-quality product left unsampled is a random variable Y with distribution $J(y)=\frac{F_D(y;M)}{F_D(p;M)}$, for any $\underline{p}_D \leq y \leq p$. Let the random variable Z be the lowest price of all the high-quality products in the next page, which is the best price in the next sample. Note that $Z=0$ when all the remaining unsampled products are low-types. When $t>0$, we have

$$Z=\begin{cases} Y_{(1)} & \text{if } 1 \leq t \leq M, \\ Y_{(t-M+1)} & \text{if } t>M, \end{cases}$$

where $Y_{(n)}$ is the $n-th$ order statistic (or, $n-th$ smallest order statistic) of the sample formed by t random variables, which are independently and identically distributed according to $J(y)$. In other words, if there are less than M high-quality products left, then the best price in the next page is just the lowest price of all remaining high-quality products. On the other hand, if there are more than M high-quality products left, then the best price is the $M-th$ largest price, or $(t-M+1)-th$ smallest price, of all the t prices.

Let $f_Z(z;t,p,k,M)$ be the density of Z, conditional on the state (p,k) and that there are t high-quality products left. Then

$$f_Z(z;t,p,k,M)=\begin{cases} t\left[1-J(z)\right]^{t-1} J'(z) & \text{if } 1 \leq t \leq M, \\ tC(t-1,M-1)\left[1-J(z)\right]^{m-1}\left[J(z)\right]^{t-M} J'(z) & \text{if } t>M, \end{cases} \tag{26}$$

where $J(z)=\frac{F_D(z;M)}{F_D(p;M)}$.

The expected benefit of sampling the next page, given the state (p,k) is thus

$$\phi_D(p,k;M)=\sum_{t=1}^{N-kM}\sigma(t;p,k,M)\int_{\underline{p}_D}^{p}(p-z)f_Z(z;t,p,k,M)dz. \tag{27}$$

The reservation price at stage k, $\hat{p}^M_{D,k}$, is defined to be the solution to $\phi_D(p,k;M)=c$, for $k=1,...,\frac{N}{M}-1$. The same analysis shows that as long as the reservation prices are increasing along with the search process, the optimal stopping rule in this case is to continue searching at stage k if and only if the state price p is higher than the reservation price $\hat{p}^M_{D,k}$, for all $k=1,...,\frac{N}{M}-1$. Finally, the property of increasing reservation prices is satisfied if the total number of pages $\frac{N}{M}$ is not large.

Similar to the case of $M=1$, we fix a group of reservation prices and solve the conditional optimal price distribution for high-type firms. Again, we first derive a high-type firm's demand, and then solve the optimal price distribution according to the constant-profit condition. Finally, the pre-given reservation prices should stand for the optimal stopping rule conditional on the obtained price distribution, which gives the equilibrium. The derivation of a high-type firm's demand is tidious and put in the Appendix.

Until now, we have solved the symmetric equilibrium under each type of price sorting. The next Proposition states that our main result on welfare comparions still holds even for $M > 1$.

Proposition 12 *With small search costs, given the type of price sorting $S \in \{R, A, D\}$, let TW_S^M, Π_S^M and CS_S^M be total welfare, industry profit and consumer surplus, respectively, when M products are observed in each sample. We have that*

$$\Pi_A^M = \Pi_D^M = \Pi_R^M.$$

Moreover, if $r > 1/2$,

$$TW_A^M > TW_D^M > TW_R^M, \text{ and } CS_A^M > CS_D^M > CS_R^M;$$

and if $r < 1/2$,

$$TW_D^M > TW_A^M > TW_R^M, \text{ and } CS_D^M > CS_A^M > CS_R^M.$$

Finally, for the case of endogenous price sorting, the market equilibrium can be derived the same way as we did when $M = 1$. Proposition 13 says that our main result on the choice of price sorting still holds when $M > 1$.

Proposition 13 *Suppose M products are observed in each sample. Then, with small search costs, if consumers can decide the type of price sorting, random price sorting is never selected in equilibrium. Specificly, consumers choose ascending price sorting if $r > 1/2$, and choose descending price sorting if $r < 1/2$.*

7 Conclusion

This paper considers price sorting in a consumer search model. We have studied both ascending and descending price sortings, which are the most commonly observed sorting options in the Web. Either ascending or descending price sorting can be applied before the sampling process. Consumers search sequentially for products with two types of qualities. We allow a fraction of consumers to have zero search costs, and all other consumers have the same positive search cost. Price dispersion exists in the unique symmetric equilibrium. We find that, when the search cost is small, using price sorting will improve both total welfare and consumer surplus, but have no impact on industry profits. Moreover, if consumers can choose the type of price sorting for their own interests, ascending price sorting (or descending price sorting, respectively) will be chosen if there are more high-quality products (or low-quality products, respectively) in the market.

Our analysis has been restricted to the case in which the cost of sampling a product is small. The situation may be different if the search cost becomes large. Take the comparisons of total welfare for example. With large search costs, searchers may accept any high-quality product, and only continue searching when low-quality products are observed. This means search never takes place under descending price sorting. On the other hand, compared to random price sorting, consumers search more often under ascending price sorting since low-quality products are always first sampled. As a result, ascending price sorting no longer saves consumers' total search costs and lowers total welfare compared to random price sorting. When the search cost becomes even higher, no search would ever take place under all types of price sorting, so that consumers always purchase the first product they sample. In this case, compared to random price sorting, total welfare is improved under descending price sorting, but decreased under ascending price sorting. Although the examples of online purchases fit the small search cost assumption, there are many other situations in which search costs are large. It will be a desirable extension to consider those situations.

As most of the search literature did, this paper considers an optimal search policy that is of a simple form. The only choice consumers have to make is whether to continue the search process. To be precise, when deciding to search on, consumers do not have to determine which product should be sampled next. This is because the search order is fully determined by firms' pricing strategies and the type of sorting. The derivation of this simple search policy is based on two assumptions: first, consumers observe neither price nor quality before a product is sampled; second, all products should be sampled in the same order as they are displayed according to the type of price sorting. The first assumption does not hold in situations where only prices are displayed in the web pages after the products are sorted. Thus, consumers not only choose whether or not the search on, but also decide which product to sample based on the price information. Likewise, without the second assumption, consumers will have to choose which page to go to whenever deciding to search on, based on the samples they have already observed. Consumers' search behaviors in those situations are more complex, but deserve studying in the future.

Finally, there are other types of sorting options available for consumers during the online purchases. For example, many commerce web sites allow consumers to sort their products by popularity or average customer review. There is no doubt that these sorting tools also play important roles in affecting consumer choices and product prices. The impacts of these sortings may be studied in the context of dynamic games, which is left to future work.

8 Appendix

In this section, we provide technical proofs for all the lemmas and propositions in the main text.

Proof of Lemma 1. It is obvious that the searcher will continue searching whenever $p_{D,n} \geq \hat{p}_{D,n}$, where $p_{D,n}$ is the observed price at stage n, for any $n = 1, ..., N-1$. We now prove by induction that it is optimal for the searcher to stop searching at stage n whenever $p_{D,n} < \hat{p}_{D,n}$. Our result is obviously true when $n = N-1$. Suppose it is true for all stages $N-1, N-2, ..., N-k$, and consider stage $N-k-1$, at which $p_{D,N-k-1} < \hat{p}_{D,N-k-1}$. If the searcher keeps sampling at stage $N-k-1$, since he will stop searching in the next stage (due to the inductive hypothesis, the increasing order of reservation prices and the decreasing order of observed prices), the total expected benefits of doing so is exactly equal to $\phi_D(p_{D,N-k-1}, N-k-1)$, which is below the search cost c since $p_{D,N-k-1} < \hat{p}_{D,N-k-1}$. Thus, the searcher would be better off if he chose to stop searching at stage $N-k-1$, which completes our proof. ■

Proof of Lemma 2. Expression (2) can be rewritten as

$$\phi_D(p,n) = \int_{\underline{p}_D}^{p} \left[\left(\frac{1-r+rF_D(x)}{1-r+rF_D(p)} \right)^{N-n} - \left(\frac{1-r}{1-r+rF_D(p)} \right)^{N-n} \right] dx.$$

Define

$$h(m;x,p) = \left(\frac{1-r+rF_D(x)}{1-r+rF_D(p)} \right)^{m} - \left(\frac{1-r}{1-r+rF_D(p)} \right)^{m}.$$

Then we have

$$\frac{\partial \phi_D(p,n)}{\partial n} = -\int_{\underline{p}_D}^{p} \frac{\partial h(m;x,p)}{\partial m}|_{m=N-n} dx.$$

Thus it is sufficient to prove that there exists a positive number $\overline{N}$ such that as long as $m < \overline{N}$, $\frac{\partial h(m;x,p)}{\partial m} > 0$ for any x and p. Note that

$$\begin{aligned} \frac{\partial h(m;x,p)}{\partial m} &= A^m \left\{ \left(1+\frac{y}{A}\right)^m \ln(A+y) - \ln A \right\} \\ &\equiv k(m;A,y), \end{aligned}$$

where $A = \frac{1-r}{1-r+rF_D(p)} \in [1-r, 1]$ and $y = \frac{rF_D(x)}{1-r+rF_D(p)} \in [0, 1-A]$. We have that

$$\begin{aligned} \frac{\partial k(m;A,y)}{\partial y} &= (A+y)^{m-1} \left[1 + m \ln(A+y)\right] \\ &> (A+y)^{m-1} \left[1 + m \ln A\right]. \end{aligned}$$

Define $\overline{N} = -\frac{1}{\ln(1-r)}$. Then, as long as $m < \overline{N}$, we have that $1 + m \ln A > 0$, where we have used the fact that $1-r \leq A \leq 1$. This means that $\frac{\partial k(m;A,y)}{\partial y} > 0$ for all $y \in [0, 1-A]$. Moreover, it is easy to verify that $k(m; A, y=0) = 0$ and $k(m; A, y = 1-A) = -A^m \ln A > 0$. Thus, we have that $\frac{\partial h(m;x,p)}{\partial m} = k(m;A,y) \geq 0$ for all m, x and p as long as $m < \overline{N}$, which completes our proof. ■

Proof of Lemma 3. First of all, $F_R(p; \hat{p}_R)$ cannot have an atom at $p = 0$ because a high-type firm can guarantee itself a positive expected profit by charging a price $p \in (0, a)$. By doing so, the high-type firm yields a positive surplus, so that all the shoppers will buy from it as long as all other firms are low-type, which happens with probability $(1-r)^{N-1}$. To prove that there are no atoms at positive prices, suppose the contrary. Then by slightly undercutting the atom, the firm can discretely increase its demand from the shoppers without losing any demand from the searchers, which obviously contradicts the optimality of $F_R(p;\hat{p}_R)$. ■

Proof of Lemma 4. For part (i), suppose $\overline{p}_R < \hat{p}_R$, then consider $p' \in (\overline{p}_R, \hat{p}_R)$. Note that p' gives the firm the same demand as $\overline{p}_R$ does, thus making a higher profit, contradiction. On the other hand, suppose $\hat{p}_R < \overline{p}_R < a$, similar to the previous argument, any price $p' \in (\overline{p}_R, a)$ is strictly better than $\overline{p}_R$, which is also impossible. Hence, either $\overline{p}_R = \hat{p}_R$ or $\overline{p}_R = a$. For part (ii), suppose otherwise, let $(\alpha, \beta) \subset [\underline{p}_R, \overline{p}_R]$ be the largest such gap. Then charging price β yields the same demand for the firm as α does. Since $\beta > \alpha$, β will make larger profits than α, contradiction. ■

Proof of Proposition 1. Proposition 1 follows immediately from the condition $\pi_R = pD(p)$ and the demand function (3). ■

Before giving a proof for Proposition 2, we find the following analyses very useful.

Equation (9) can be rewritten as

$$r\int_{\underline{p}_R}^{\hat{p}_R} F_R^*(x;\hat{p}_R)dx = c. \tag{28}$$

Expressions (5)-(8) imply that all $F_R^*(x;\hat{p}_R), \underline{p}_R, \hat{p}_R$ and p_R' can be expressed as functions of $\hat{F}_R$. Thus, the left-hand side of (28) depends only on $\hat{F}_R$, which in turn can be expressed as a function of c. Obviously, as $\hat{F}_R \to 0$, all $\underline{p}_R, \hat{p}_R$ and $p_R' \to \pi_R$, so that $r\int_{\underline{p}_R}^{\hat{p}_R} F_R^*(x;\hat{p}_R)dx \to 0$. Moreover, due to (5)-(8), as $\hat{F}_R \to 0$, we have that $\frac{\partial \underline{p}_R}{\partial \hat{F}_R} \to \frac{1}{2}(N-1)\pi_R r(1-\mu)$, $\frac{\partial \hat{p}_R}{\partial \hat{F}_R} \to \frac{1}{2}(N-1)\pi_R r(1+\mu)$, $\frac{\partial p_R'}{\partial \hat{F}_R} \to (N-1)\pi_R r$, and $\frac{\partial F_R^*(x;\hat{p}_R)}{\partial \hat{F}_R} \to -\frac{1-\mu}{2\mu}$ for any $x \in [\underline{p}_R, \hat{p}_R]$.

Taking derivative of the left-hand side of (28) with respect to $\hat{F}_R$ gives

$$\begin{aligned}\frac{\partial \int_{\underline{p}_R}^{\hat{p}_R} F_R^*(x;\hat{p}_R)dx}{\partial \hat{F}_R} &= \hat{F}_R\frac{\partial \hat{p}_R}{\partial \hat{F}_R} + \int_{\underline{p}_R}^{\hat{p}_R} \frac{\partial F_R^*(x;\hat{p}_R)}{\partial \hat{F}_R}dx \\ &= \hat{F}_R\left[\frac{\partial \hat{p}_R}{\partial \hat{F}_R} + \frac{\int_{\underline{p}_R}^{\hat{p}_R} \frac{\partial F_R^*(x;\hat{p}_R)}{\partial \hat{F}_R}dx}{\hat{F}_R}\right].\end{aligned}$$

Since

$$\lim_{\hat{F}_R\to 0}\frac{\partial \hat{p}_R}{\partial \hat{F}_R} = \frac{1}{2}(N-1)\pi_R r(1+\mu)$$

and

$$\begin{aligned}\lim_{\hat{F}_R\to 0}\frac{\int_{\underline{p}_R}^{\hat{p}_R} \frac{\partial F_R^*(x;\hat{p}_R)}{\partial \hat{F}_R}dx}{\hat{F}_R} &= \lim_{\hat{F}_R\to 0}\frac{-(1-\mu)}{2\mu}\left(\frac{\partial \hat{p}_R}{\partial \hat{F}_R} - \frac{\partial \underline{p}_R}{\partial \hat{F}_R}\right) \\ &= -\frac{1}{2}(N-1)\pi_R r(1-\mu),\end{aligned}$$

the above expression implies that

$$\lim_{\hat{F}_R\to 0}\frac{\partial\int_{\underline{p}_R}^{\hat{p}_R}F_R^*(x;\hat{p}_R)dx}{\partial\hat{F}_R}\frac{1}{\hat{F}_R}=(N-1)\,\pi_R r\mu. \tag{29}$$

Thus, when $\hat{F}_R$ is sufficiently small, the left-hand side of (28) is strictly increasing in $\hat{F}_R$. This means that, there is always a unique $\hat{F}_R$ which solves Equation (28) for small search costs.

The above useful results are summarized as follows.

Result 1: As $c\to 0$, we have that
(i)

$$\begin{aligned}\frac{\partial\underline{p}_R}{\partial\hat{F}_R}&\to\frac{1}{2}(N-1)\,\pi_R r\,(1-\mu)\,,\\ \frac{\partial\hat{p}_R}{\partial\hat{F}_R}&\to\frac{1}{2}(N-1)\,\pi_R r\,(1+\mu)\,,\\ \frac{\partial p_R'}{\partial\hat{F}_R}&\to(N-1)\,\pi_R r;\end{aligned}$$

(ii) when $x\in[\underline{p}_R,\hat{p}_R]$, we have

$$\frac{\partial F_R^*(x;\hat{p}_R)}{\partial\hat{F}_R}\to-\frac{1-\mu}{2\mu};$$

(iii) finally, we have

$$\begin{aligned}\frac{\partial\hat{F}_R}{\partial c}\hat{F}_R&\to\frac{1}{(N-1)\,\pi_R r^2\mu},\\ \frac{c}{\hat{F}_R^2}&\to\frac{(N-1)\,\pi_R r^2\mu}{2},\\ \frac{c}{\hat{F}_R}&\to 0,\\ \frac{\partial\hat{F}_R}{\partial c}c&\to 0.\end{aligned}$$

Proof. Parts (i) and (ii) are obvious. For (iii), taking derivatives of both sides of (28) with respect to c gives

$$r\hat{F}_R\frac{\partial\hat{p}_R}{\partial\hat{F}_R}\frac{\partial\hat{F}_R}{\partial c}+r\frac{\partial\hat{F}_R}{\partial c}\int_{\underline{p}_R}^{\hat{p}_R}\frac{\partial F_R^*(x;\hat{p}_R)}{\partial\hat{F}_R}dx=1,$$

or

$$r\frac{\partial\int_{\underline{p}_R}^{\hat{p}_R}F_R^*(x;\hat{p}_R)dx}{\partial\hat{F}_R}\frac{\partial\hat{F}_R}{\partial c}=1.$$

According to (29), the above expression gives the first result:

$$\frac{\partial \hat{F}_R}{\partial c}\hat{F}_R \to \frac{1}{(N-1)\,\pi_R r^2 \mu}.$$

For the second result, note that

$$\begin{aligned}
\lim_{c\to 0}\frac{c}{\hat{F}_R^2} &= \lim_{c\to 0}\frac{r\int_{\underline{p}_R}^{\hat{p}_R} F_R^*(x;\hat{p}_R)dx}{\hat{F}_R^2} \\
&= r\lim_{\hat{F}_R\to 0}\frac{\partial\int_{\underline{p}_R}^{\hat{p}_R} F_R^*(x;\hat{p}_R)dx/\partial\hat{F}_R}{2\hat{F}_R} \\
&= \frac{r}{2}\lim_{\hat{F}_R\to 0}\frac{\hat{F}_R\frac{\partial\hat{p}_R}{\partial\hat{F}_R}+\int_{\underline{p}_R}^{\hat{p}_R}\frac{\partial F_R^*(x;\hat{p}_R)}{\partial\hat{F}_R}dx}{\hat{F}_R} \\
&= \frac{r}{2}\lim_{\hat{F}_R\to 0}\left[\frac{\partial\hat{p}_R}{\partial\hat{F}_R}+\frac{\int_{\underline{p}_R}^{\hat{p}_R}\frac{\partial F_R^*(x;\hat{p}_R)}{\partial\hat{F}_R}dx}{\hat{F}_R}\right] \\
&= \frac{(N-1)\,\pi_R r^2\mu}{2},
\end{aligned}$$

where the second equality is due to L'Hospital rule.

Finally, the third and fourth results of part (iii) are direct consequences of the the first two since $\lim_{c\to 0}\hat{F}_R = 0$. ■

Proof of Proposition 2. The above analysis has shown that the solution of Equation (9) always exists and is unique for small search costs. In addition, the equilibrium reservation price $\hat{p}_R$ increases with the search cost because $\frac{\partial p_R}{\partial c} = \frac{\partial p_R}{\partial \hat{F}_R}\frac{\partial \hat{F}_R}{\partial c} > 0$ due to Result 1. ■

Proof of Lemma 5. The proof is similar to that of Lemma 3. Firstly, the equilibrium price distribution cannot have an atom at price zero because each high-type firm's expected profit is strictly positive. Secondly, to prove that there is no atom at any positive price, suppose the contrary. Then by slightly undercutting the atom, a high-type firm not only discretely increases its demand from the shoppers, but also increases its demand from the searchers because a lower price means being sampled earlier and having a higher surplus. Thus, the high-type firm can have a strictly higher profit by doing so, which obviously contradicts the optimality of $F_A(p)$. ■

Proof of Lemma 6. The proof is similar to that of Lemma 4. Suppose $\overline{p}_A < a$, then any price $p' \in (\overline{p}_A, a)$ gives a high-type firm the same demand as $\overline{p}_A$ does, thus making a higher profit, contradiction. To show that there is no gap in the equilibrium price support, suppose otherwise and let $(\alpha,\beta) \subset [\underline{p}_A, \overline{p}_A]$ be the largest such gap. Then charging price β yields the same demand for the firm as α does. Since $\beta > \alpha$, β will make larger profits than α, contradiction. ■

Proof of Proposition 3. When $c < \phi_A$, searchers never stop sampling until a high-quality product is observed. Due to the increasing order of prices, each searcher ends up purchasing the high-quality product with the lowest price, and behaves exactly the same as a shopper. Thus, Expressions (12) and (13) follow immediately from the condition $\pi_A = pD(p)$ and the demand function (10). ■

Proof of Lemma 7. Suppose all high-type firms set the price $p = a$. Thus, both types of firms offer a surplus of zero, and searchers will never sample the next product. The expected profit of a high-type firm is given by

$$\pi_D(a) = a \left\{ \frac{\mu}{N} + \frac{(1-\mu)\left[1-(1-r)^N\right]}{Nr} \right\},$$

where $\frac{\mu}{N}$ and $\frac{(1-\mu)\left[1-(1-r)^N\right]}{Nr}$ are the demands from shoppers and searchers, respectively. To understand the demand from searchers, note that with probability $C(N-1,k)r^k(1-r)^{N-1-k}$, there are k other high-type firms in the market. In this case, all the $(k+1)$ high-type firms set price $p = a$, and other low-type firms set zero prices. Due to the decreasing order of prices, each searcher randomly purchases from the $(k+1)$ high-type firms. Thus, the high-type firm's total demand from searchers is given by

$$\begin{aligned}
& (1-\mu)\sum_{k=0}^{N-1} C(N-1,k)r^k(1-r)^{N-1-k}\frac{1}{k+1} \\
= \; & (1-\mu)\sum_{k=0}^{N-1} \frac{(N-1)!}{(N-1-k)!k!}r^k(1-r)^{N-1-k}\frac{1}{k+1} \\
= \; & \frac{(1-\mu)}{rN}\sum_{k=0}^{N-1} \frac{N!}{(N-1-k)!\,(k+1)!}r^{k+1}(1-r)^{N-1-k} \\
= \; & \frac{(1-\mu)}{rN}\sum_{k=0}^{N-1} C(N,k+1)r^{k+1}(1-r)^{N-1-k} \\
= \; & \frac{(1-\mu)}{rN}\left[1-(1-r)^N\right].
\end{aligned}$$

Now we check that for high-type firms, any price deviation $p < a$ is not profitable. Given that all other high-type firms set price $p = a$, a high-type firm that charges $p < a$ will earn a profit of

$$\pi_D(p) = p[\mu + (1-\mu)(1-r)^{N-1}].$$

We have that

$$\begin{aligned}\pi_D(a)-\pi_D(p) &> a\left\{\frac{\mu}{N}+\frac{(1-\mu)\left[1-(1-r)^N\right]}{Nr}\right\} \\ &\quad -a[\mu+(1-\mu)(1-r)^{N-1}] \\ &= A\left[\frac{1-(1-r)^N-rN\,(1-r)^{N-1}}{1-(1-r)^N-rN\,(1-r)^{N-1}+r(N-1)}-\mu\right],\end{aligned}$$

where $A=\frac{a\left[1-(1-r)^N-rN(1-r)^{N-1}+r(N-1)\right]}{rN}>0$.

Obviously, if $\mu\leq\frac{1-(1-r)^N-rN(1-r)^{N-1}}{1-(1-r)^N-rN(1-r)^{N-1}+r(N-1)}$, then $\pi_D(a)-\pi_D(p)>0$ so that any $p<a$ is not a profitable deviation, which completes our proof. ■

Proof of Lemma 8. The key point is that we focus on the small-search-cost situations so that search takes place at each stage as long as the observed price is high enough. Thus, the upper bound of price support $\overline{p}_D$ should be above the highest reservation price $\hat{p}_{D,N-1}$. This means, charging $\overline{p}_D$ gives a high-type firm a demand $(1-r)^{N-1}$ because consumers will purchase from this firm if and only if it is the only high-type firm in the market. Suppose $\overline{p}_D<a$, since any high-type firm can have the same demand $(1-r)^{N-1}$ by charging price a, increasing the price from $\overline{p}_D$ to a becomes a strictly profitable deviation. ■

Proof of Proposition 4. Proposition 4 follows immediately from the demand function (18) and the condition that $\pi_D=pD(p)$ for all p. ■

According to Proposition 4, the optimal price distribution function $F_D^*(p;\hat{p}_D)$ and the elements $\underline{p}_D$, $\hat{p}_{D,n}$ and $p'_{D,n}$ can be expressed as functions of $\hat{F}_{D,1}$, $\hat{F}_{D,2}$,..., $\hat{F}_{D,N-1}$, for $n=1,...,N-1$. More precisely, for any n and $p\in(\hat{p}_{D,n},\hat{p}_{D,n+1}]$, $F_D(p;\hat{p}_D)$ is the solution of the following equation.

$$\begin{aligned}\frac{\pi_D}{p} &= \mu\left[1-rF_D(p;\hat{p}_D)\right]^{N-1} \\ &+(1-\mu)\sum_{k=0}^{n-1}C(N-1,k)\,(1-r)^{N-1-k}\left[r-rF_D(p;\hat{p}_D)\right]^k \\ &+(1-\mu)\,C(N-1,n)\,(1-r+rF_D(p;\hat{p}_D))^{N-1-n}\left[r-rF_D(p;\hat{p}_D)\right]^n \\ &+(1-\mu)\sum_{k=n+1}^{N-1}C(N-1,k)\,(1-r+rF_D(p;\hat{p}_D))^{N-1-k}\left[r-r\hat{F}_{D,k}\right]^k,\end{aligned}\tag{30}$$

moreover, we have that

$$\frac{\pi_D}{\underline{p}_D}=\mu+(1-\mu)\sum_{k=0}^{N-1}C(N-1,k)\,(1-r)^{N-1-k}\left[r-r\hat{F}_{D,k}\right]^k,\tag{31}$$

$$
\begin{aligned}
\frac{\pi_D}{p'_{D,n}} &= \mu\left[1 - r\hat{F}_{D,n}\right]^{N-1} + (1-\mu)\sum_{k=0}^{n-1} C(N-1,k)\,(1-r)^{N-1-k}\left[r - r\hat{F}_{D,n}\right]^{k} \\
&+ (1-\mu)\,C(N-1,n)\left(1-r+r\hat{F}_{D,n}\right)^{N-1-n}\left[r - r\hat{F}_{D,n}\right]^{n} \\
&+ (1-\mu)\sum_{k=n+1}^{N-1} C(N-1,k)\left(1-r+r\hat{F}_{D,n}\right)^{N-1-k}\left[r - r\hat{F}_{D,k}\right]^{k},
\end{aligned}
\tag{32}
$$

$$
\begin{aligned}
\frac{\pi_D}{\hat{p}_{D,n}} &= \mu\left[1 - r\hat{F}_{D,n}\right]^{N-1} + (1-\mu)\sum_{k=0}^{n-1} C(N-1,k)\,(1-r)^{N-1-k}\left[r - r\hat{F}_{D,n}\right]^{k} \\
&+ (1-\mu)\,C(N-1,n)\left(1-r+r\hat{F}_{D,n}\right)^{N-1-n}\left[r - r\hat{F}_{D,n}\right]^{n} \\
&+ (1-\mu)\sum_{k=n+1}^{N-1} C(N-1,k)\left(1-r+r\hat{F}_{D,n}\right)^{N-1-k}\left[r - r\hat{F}_{D,k}\right]^{k} \\
&+ (1-\mu)\,C(N-1,n-1)\left(r - r\hat{F}_{D,n}\right)^{n-1}\left[\left(1-r+r\hat{F}_{D,n}\right)^{N-n} - (1-r)^{N-n}\right].
\end{aligned}
\tag{33}
$$

Using the above expressions, it is not difficult to verify the following results.

Result 2: As $\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1} \to 0$, we have that,
(i) for any $n = 1, ..., N-1$,

$$
\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n}} \to \pi_D\,(1-\mu)\,nC(N-1,n)\,(1-r)^{N-1-n}\,r^{n};
\tag{34}
$$

(ii) for any $n = 1, ..., N-1$ and $m \geq n+1$,

$$
\frac{\partial p'_{D,n}}{\partial \hat{F}_{D,n}} \;\to\; \pi_D r\,(N-1)\left[1 - 2\,(1-\mu)\sum_{k=n+1}^{N-1} C(N-2,k-1)\,(1-r)^{N-k-1}\,r^{k-1}\right],
\tag{35}
$$

$$
\frac{\partial p'_{D,n}}{\partial \hat{F}_{D,m}} \;\to\; \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,m}} \to \pi_D\,(1-\mu)\,mC(N-1,m)\,(1-r)^{N-1-m}\,r^{m};
\tag{36}
$$

(iii) for any $n = 1, ..., N-1$ and $m \geq n+1$,

$$\begin{aligned}
\frac{\partial \hat{p}_{D,n}}{\partial \hat{F}_{D,n}} &\rightarrow \frac{\partial p'_{D,n}}{\partial \hat{F}_{D,n}} - \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n}} \\
&\rightarrow \pi_D r (N-1) \left[1 - 2(1-\mu) \sum_{k=n+1}^{N-1} C(N-2, k-1) (1-r)^{N-k-1} r^{k-1} \right] \\
&\quad -\pi_D (1-\mu) n C(N-1, n) (1-r)^{N-1-n} r^n, \qquad (37) \\
\frac{\partial \hat{p}_{D,n}}{\partial \hat{F}_{D,m}} &\rightarrow \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,m}} \rightarrow \pi_D (1-\mu) m C(N-1, m) (1-r)^{N-1-m} r^m; \qquad (38)
\end{aligned}$$

(iv) for any $n = 1, ..., N-1$,

$$\frac{\partial \hat{p}_{D,n+1}}{\partial \hat{F}_{D,n+1}} - \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n+1}} \rightarrow \frac{\partial p'_{D,n}}{\partial \hat{F}_{D,n}}. \qquad (39)$$

(v) for any n, any $p \in (\hat{p}_{D,n}, \hat{p}_{D,n+1}]$ and $m \geq n+1$, we have

$$\frac{\partial F_D^*(p; \hat{p}_D)}{\partial \hat{F}_{D,m}} \rightarrow -\frac{\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,m}}}{\frac{\partial p'_{D,n}}{\partial \hat{F}_{D,n}}}. \qquad (40)$$

According to (39), for any $p \in (\hat{p}_{D,n}, \hat{p}_{D,n+1}]$, we have

$$\frac{\partial F_D^*(p; \hat{p}_D)}{\partial \hat{F}_{D,n+1}} \rightarrow -\frac{\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n+1}}}{\frac{\partial p'_{D,n}}{\partial \hat{F}_{D,n}}} \rightarrow \frac{-\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n+1}}}{\frac{\partial \hat{p}_{D,n+1}}{\partial \hat{F}_{D,n+1}} - \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n+1}}}. \qquad (41)$$

Note that (23) gives an equation system with $N-1$ equations and $N-1$ variables $(\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1})$. For any $n = 1, ..., N-1$, define

$$\Psi_n(\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1}) \equiv \int_{\underline{p}_D}^{\hat{p}_{D,n}} (\hat{p}_{D,n} - x) d \left(\frac{1 - r + r F_D^*(x; \hat{p}_D)}{1 - r + r \hat{F}_{D,n}} \right)^{N-n}.$$

Taking derivatives of both sides of (23) with respect to c gives that

$$\sum_{m=1}^{N-1} \frac{\partial \Psi_n}{\partial \hat{F}_{D,m}} \frac{\partial \hat{F}_{D,m}}{\partial c} = 1 \text{ for all } n. \qquad (42)$$

Using Result 2, it is not difficult to find that, for any $n, m = 1, ..., N-1$, as $\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1} \rightarrow$

0,

$$\frac{\partial \Psi_n}{\partial \hat{F}_{D,m}} \frac{1}{\hat{F}_{D,m}} \to \begin{cases} -\frac{r(N-n)}{1-r} \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,m}} & \text{if } m < n; \\ \frac{r(N-n)}{1-r} \frac{\partial \left(\hat{p}_{D,m} - \underline{p}_D\right)}{\partial \hat{F}_{D,m}} & \text{if } m = n; \\ \frac{r(N-n)}{1-r} \frac{\partial \underline{p}_D}{\partial \hat{F}_{D,m}} & \text{if } m > n. \end{cases} \tag{43}$$

For notational ease, we use $\hat{F}_D \to 0$ to represent $\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1} \to 0$. Define

$$\begin{aligned} a_{n,m} &\equiv \lim_{\hat{F}_D \to 0} \frac{\partial \Psi_n}{\partial \hat{F}_{D,m}} \frac{1}{\hat{F}_{D,m}}, \\ b_m &\equiv \lim_{\hat{F}_D \to 0} \frac{\partial \hat{F}_{D,m}}{\partial c} \hat{F}_{D,m}, \end{aligned}$$

then (42) means

$$\sum_{m=1}^{N-1} a_{n,m} b_m = 1 \text{ for all } n. \tag{44}$$

Let matrix A be the $(N-1) \times (N-1)$ square matrix with $a_{n,m}$ as the element in the $n-th$ row and $m-th$ column. According to (43), it is not difficult to verify that matrix A is invertible so that there exists a unique set of $(b_1, ..., b_{N-1})$ that solves (44), which in turn proves that the solution of (23), $\left(\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1}\right)$, uniquely exists when the search cost c is approaching zero.

Proof of Proposition 5. Since $(\hat{p}_{D,1}, ..., \hat{p}_{D,N-1})$ are functions of $\left(\hat{F}_{D,1}, \hat{F}_{D,2}, ..., \hat{F}_{D,N-1}\right)$, the above analysis simply shows that there exists a unique set of reservation prices $(\hat{p}_{D,1}, ..., \hat{p}_{D,N-1})$ that solves Equation (23). ■

Proof of Proposition 6. Under random price sorting, there is probability $r\hat{F}_R$ that each sampled product is a Type I product, that is, the product is of high quality and with a price lower than the reservation price, in which case the searcher stops sampling and make a purchase immediately. There is probability $1 - r\hat{F}_R$ that each sampled product is a Type II product, that is, the product is either of low quality, or of high quality but with a price higher than the reservation price, in which case the searcher will continue searching.

Thus, the probability that m searches take place[19] is $\left(1 - r\hat{F}_R\right)^m r\hat{F}_R$, for $m = 0, 1, ..., N-2$. This is because this happens when the first m samples are of Type II, and the $(m+1)\,th$ sample is of Type I. Finally, the probability that $N-1$ searches take place is $\left(1 - r\hat{F}_R\right)^{N-1}$. This is because a searcher sample all the products if and only if all the first $N-1$ samples are of Type II.

[19] We have assumed that the first sample is free. And the total number of searches we are exploring for each type of sorting is defined to be the number of samples other than the first one.

Then the expected number of searches under random price sorting is given by

$$\begin{aligned} N_R &= \sum_{m=1}^{N-2} m\left(1-r\hat{F}_R\right)^m r\hat{F}_R + (N-1)\left(1-r\hat{F}_R\right)^{N-1} \\ &= \frac{1-\left(1-r\hat{F}_R\right)^N}{r\hat{F}_R} - 1, \end{aligned}$$

which converges to $N-1$ as $c \to 0$.

Under ascending price sorting, for small search costs, a searcher will sample all the low-quality products and purchse the high-quality product with the lowest price. So the probability that a searcher makes m samples[20] is $C(N,m)\,(1-r)^m\, r^{N-m}$ because it happens when m out of N products are of low quality, for $m = 0, 1, ..., N-2$. The probability that the searcher makes $N-1$ samples (i.e., sample all the products) is $C(N, N-1)\,(1-r)^{N-1}\, r + (1-r)^N$ because it happens when there are $N-1$ or N low-quality products. Thus, the expected number of searches under ascending price sorting is

$$\begin{aligned} N_A &= \sum_{m=1}^{N-1} mC(N,m)\,(1-r)^m\, r^{N-m} + (N-1)\,(1-r)^N \\ &= \sum_{m=1}^{N} mC(N,m)\,(1-r)^m\, r^{N-m} - (1-r)^N \\ &= N\,(1-r)\sum_{m=1}^{N} C(N-1,m-1)\,(1-r)^{m-1}\, r^{N-m} - (1-r)^N \\ &= N\,(1-r) - (1-r)^N\,, \end{aligned}$$

where the third equality comes from the fact that $mC(N,m) = NC(N-1,m-1)$ and the last comes from the fact that $\sum_{m=1}^{N} C(N-1,m-1)\,(1-r)^{m-1}\, r^{N-m} = 1$.

Finally, under descending price sorting, let $p_{(n)}$ be the $n-th$ highest price out of all N prices. Given the reservation prices $0 < \hat{p}_{D,1} \leq \hat{p}_{D,2} \leq ... \leq \hat{p}_{D,N-1} < a$ and the optimal stopping rule, for any $m = 1, 2, ..., N-2$, the searcher makes m samples if and only if $p_{(m)} > \hat{p}_{D,m}$ and $p_{(m+1)} < \hat{p}_{D,m+1}$. On the other hand, $N-1$ searches take place if and only if $p_{(N-1)} > \hat{p}_{D,N-1}$. Define α_m to be the probability that m searches take place, for $m = 1, 2, ..., N-1$. Then, as the search cost $c \to 0$, $\hat{p}_{D,n} \to \underline{p}_D$ for all n, and hence we have that $\alpha_m \to C(N,m)\,(1-r)^{N-m}\, r^m$ for $m = 1, 2, ..., N-2$ and $\alpha_{N-1} \to C(N, N-1)\,(1-r)\, r^{N-1} + r^N$. The expected number of searches under descending price sorting is

$$\begin{aligned} \lim_{c\to 0} N_D &= \sum_{m=1}^{N-1} mC(N,m)\,(1-r)^{N-m}\, r^m + (N-1)\, r^N \\ &= Nr - r^N, \end{aligned}$$

[20] The first sample is not counted.

where the last equality follows the same way as that of the derivation of N_A.

Define $h(r) \equiv Nr - r^N$, for all $r \in [0,1]$. The above results can be summarized as

$$\lim_{c\to 0} N_R = h(1), \;\; \lim_{c\to 0} N_A = h(1-r), \;\text{ and }\; \lim_{c\to 0} N_D = h(r).$$

It is obvious that $h'(r) > 0$. This implies that

$$h(1) > h(r) \text{ and } h(1) > h(1-r)$$

and

$$h(r) > h(1-r) \text{ if and only } r > 1/2,$$

which completes our proof. ■

Result 3: Given any high-type firm's price distribution, when search cost c is small, compared to random price sorting, consumers search less often under both ascending and descending price sortings. Moreover, consumers search least frequently under ascending (or descending, respectively) price sorting if $r > 1/2$ (or $r < 1/2$, respectively).

Proof. The proof is the similar to that of Proposition 6. The basic logic is that, given the probability r, as the search cost c approaches zero, the expected number of searches under random price sorting is always $N-1$. In other words, searchers will sample all the firms in the market. However, under ascending price sorting, the expected number of searches is the expected number of low-type firms in the market, $N(1-r) - (1-r)^N$. This is because searchers stop sampling only after all the low-type firms are sampled. Similarly, under descending price sorting, the expected number of searches is equal to the expected number of high-type firms in the market, $Nr - r^N$, because searchers stop when they just finishing sampling all the high-type firms. Thus, the comparison of the total number of searches follows exactly the same way as that in Proposition 6. ■

Proof of Proposition 7. When search costs are small, both shoppers and searchers always end up purchasing a high-quality product as long as there is at least one high-type firm in the market. This means that the surplus of trade under each type of price sorting is the same and equal to $a\left[1-(1-r)^N\right]$. The total welfare under price sorting $S \in \{R, A, D\}$ is given by

$$TW_S = a\left[1-(1-r)^N\right] - cN_S.$$

Hence, Proposition 7 follows immediately from Proposition 6. ■

Proof of Proposition 8. Proposition 8 follows from the facts that $\pi_R = \pi_A = \pi_D = a(1-r)^{N-1}$ as long as the search cost is small, and that low-type firms always earn zero profit, regardless of the types of price sorting. ■

Proof of Proposition 9. Proposition 9 simply follows Proposition 7 and 8. ∎

Proof of Proposition 10. Since $CS_S = \mu CS_S^0 + (1-\mu) CS_S^1$ for all $S \in \{R, A, D\}$, using Proposition 6, it suffices to show that

$$\lim_{c\to 0} \frac{CS_A^0 - CS_R^0}{c} = 0 \text{ and } \lim_{c\to 0} \frac{CS_D^0 - CS_R^0}{c} = 0.$$

Since CS_A^0 does not depend on c (for small c) and

$$\lim_{c\to 0} CS_R^0 = \lim_{c\to 0} CS_D^0 = CS_A^0,$$

it suffices to show that

$$\lim_{c\to 0} \frac{\partial CS_R^0}{\partial c} = 0 \text{ and } \lim_{c\to 0} \frac{\partial CS_D^0}{\partial c} = 0.$$

The shopper's surplus under random price sorting is given as follows.

$$\begin{aligned} CS_R^0 &= \sum_{m=1}^{N} C(N,m)(1-r)^{N-m} r^m \int_{\underline{p}_R}^{a} (a-x)\, dG_{R,m}(x) \\ &= \sum_{m=1}^{N} C(N,m)(1-r)^{N-m} r^m \int_{\underline{p}_R}^{a} G_{R,m}(x) dx, \end{aligned}$$

where $G_{R,m}(x) = 1 - [1 - F_R^*(x)]^m$ represents the cumulative distribution function of the smallest order statistic with m samples, since shoppers always purchase from the lowest-priced firm. To understand the above expression, note that the typical term of the right-hand side

$$C(N,m)(1-r)^{N-m} r^m \int_{\underline{p}_R}^{a} (a-x)\, dG_{R,m}(x)$$

represents the shopper's expected surplus when there are m high-type firms in the market.

Using Result 1, one can easily derive that, for any $m = 1, ..., N$, as $\hat{F}_R \to 0$,

$$\begin{aligned} \frac{\partial \int_{\underline{p}_R}^{a} G_{R,m}(x)dx}{\partial c} &= \int_{\underline{p}_R}^{a} \frac{\partial G_{R,m}(x)}{\partial c} dx \\ &= \frac{\partial \hat{F}_R}{\partial c} \int_{\underline{p}_R}^{\hat{p}_R} m\,[1 - F_R^*(x)]^{m-1} \frac{\partial F_R^*(x)}{\partial \hat{F}_R} dx \\ &\quad + \frac{\partial \hat{F}_R}{\partial c} (p_R' - \hat{p}_R) \frac{\partial \left[1 - \left(1 - \hat{F}_R\right)^m\right]}{\partial \hat{F}_R} \\ &\to \frac{\partial \hat{F}_R}{\partial c} \hat{F}_R \{-\frac{\hat{p}_R - \underline{p}_R}{\hat{F}_R} \frac{m(1-\mu)}{2\mu} + \frac{m(p_R' - \hat{p}_R)}{\hat{F}_R}\} \\ &\to 0, \end{aligned}$$

which implies that $\lim_{c\to 0} \frac{\partial CS_R^0}{\partial c} = 0$.

The shopper's surplus under descending price sorting can be derived in the similar way as that under random price sorting. We have that

$$\begin{aligned} CS_D^0 &= \sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m \int_{\underline{p}_D}^{a} (a-x)\,dG_{D,m}(x) \\ &= \sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m \int_{\underline{p}_D}^{a} G_{D,m}(x)dx, \end{aligned}$$

where $G_{D,m}(x) = 1 - [1 - F_D^*(x)]^m$. It suffices to show that, as $c \to 0$, $\frac{\partial \int_{\underline{p}_D}^{a} G_{D,m}(x)dx}{\partial c} \to 0$ for any m.

Using Result 2, one can verify that for any $m = 1, ..., N$,

$$\begin{aligned} \frac{\partial \int_{\underline{p}_D}^{a} G_{D,m}(x)dx}{\partial c} &= \int_{\underline{p}_D}^{a} \frac{\partial G_{D,m}(x)}{\partial c} dx \\ &= \sum_{n=1}^{N-1} \int_{p'_{D,n-1}}^{\hat{p}_{D,n}} m\,[1 - F_D^*(x)]^{m-1} \frac{\partial F_D^*(x)}{\partial c} dx \\ &\quad + \sum_{n=1}^{N-1} \frac{\partial \hat{F}_{D,n}}{\partial c} \left(p'_{D,n} - \hat{p}_{D,n}\right) m \left[1 - \hat{F}_{D,n}\right]^{m-1}. \end{aligned}$$

As $c \to 0$, we have that

$$\begin{aligned} \frac{\partial \int_{\underline{p}_D}^{a} G_{D,m}(x)dx}{\partial c} &\to \sum_{n=1}^{N-1} \int_{p'_{D,n-1}}^{\hat{p}_{D,n}} m\,[1 - F_D^*(x)]^{m-1} \left[\sum_{m=n}^{N-1} \frac{\partial F_D^*(x)}{\partial \hat{F}_{D,m}} \frac{\partial \hat{F}_{D,m}}{\partial c}\right] dx \\ &\quad + \sum_{n=1}^{N-1} \frac{\partial \hat{F}_{D,n}}{\partial c} m \left(p'_{D,n} - \hat{p}_{D,n}\right) \\ &\to \sum_{n=1}^{N-1} \frac{m\left(\hat{p}_{D,n} - p'_{D,n-1}\right)}{\hat{F}_{D,n}} \left[\sum_{m=n}^{N-1} \frac{\partial F_D^*(p'_{D,n-1})}{\partial \hat{F}_{D,m}} \frac{\partial \hat{F}_{D,m}}{\partial c} \hat{F}_{D,n}\right] \\ &\quad + \sum_{n=1}^{N-1} \frac{\partial \hat{F}_{D,n}}{\partial c} m \left(p'_{D,n} - \hat{p}_{D,n}\right) \\ &\to \sum_{n=1}^{N-1} \frac{m\left(\hat{p}_{D,n} - p'_{D,n-1}\right)}{\hat{F}_{D,n}} \frac{\partial F_D^*(p'_{D,n-1})}{\partial \hat{F}_{D,n}} \frac{\partial \hat{F}_{D,n}}{\partial c} \hat{F}_{D,n} \\ &\quad + \sum_{n=1}^{N-1} \frac{\partial \hat{F}_{D,n}}{\partial c} \hat{F}_{D,n} \frac{m\left(p'_{D,n} - \hat{p}_{D,n}\right)}{\hat{F}_{D,n}}. \end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial \int_{\underline{p}_D}^{a} G_{D,m}(x)dx}{\partial c} &\to \sum_{n=1}^{N-1}\left[\frac{\hat{p}_{D,n}-p'_{D,n-1}}{\hat{F}_{D,n}}\frac{\partial F_D^*(p'_{D,n-1})}{\partial \hat{F}_{D,n}}+\frac{p'_{D,n}-\hat{p}_{D,n}}{\hat{F}_{D,n}}\right] m\frac{\partial \hat{F}_{D,n}}{\partial c}\hat{F}_{D,n}\\
&\to \sum_{n=1}^{N-1}\left[\frac{\partial\left(\hat{p}_{D,n}-p'_{D,n-1}\right)}{\partial \hat{F}_{D,n}}\frac{\partial F_D^*(p'_{D,n-1})}{\partial \hat{F}_{D,n}}+\frac{\partial\left(p'_{D,n}-\hat{p}_{D,n}\right)}{\partial \hat{F}_{D,n}}\right] m\frac{\partial \hat{F}_{D,n}}{\partial c}\hat{F}_{D,n}\\
&\to \sum_{n=1}^{N-1}\left[\frac{\partial\left(\hat{p}_{D,n}-\underline{p}_D\right)}{\partial \hat{F}_{D,n}}\frac{\partial F_D^*(p'_{D,n-1})}{\partial \hat{F}_{D,n}}+\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n}}\right] m\frac{\partial \hat{F}_{D,n}}{\partial c}\hat{F}_{D,n}\\
&\to 0,
\end{aligned}$$

where $p'_{D,0} \equiv \underline{p}_D$ and the last step follows from the fact that

$$\frac{\partial F_D^*(p'_{D,n-1})}{\partial \hat{F}_{D,n}} \to \frac{-\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n}}}{\frac{\partial \hat{p}_{D,n}}{\partial \hat{F}_{D,n}}-\frac{\partial \underline{p}_D}{\partial \hat{F}_{D,n}}},$$

which is due to (41). Thus, we have proved that $\lim_{c\to 0}\frac{\partial CS_D^0}{\partial c}=0$, which completes the proof of Proposition 10. ■

Proof of Proposition 11. For part (i), given that the search cost is approaching zero, and that all high-type firms are following the optimal price distribution $F_R(p)$, searchers can always become strictly better off by switching from random price sorting to ascending price sorting. Firstly, by doing so searchers can maximize their purchase surplus because they end up paying the lowest price. Secondly, ascending price sorting saves the total search costs because the total expected number of searches is smaller under ascending price sorting than under random price sorting, as is shown in Result 3. Thus, random price sorting is never part of equilibrium.

For part (ii), the analysis is similar to that for part (i). When $r > 1/2$, according to Result 3, as the search cost approaches zero, ascending price sorting is the best sorting option for searchers for any given price distributions. This is because it not only gives the highest purchase surplus (searchers behave like shoppers), but also gives the lowest total search cost (searchers search least frequently). This proves that the only equilibrium should be $(F_A(p), A)$.

For part (iii), suppose $r < 1/2$, and the search cost approaches zero. To prove that $(F_D(p), D)$ is the unique equilibrium, we have to prove two statements: (1) given the high-type firm's price distribution $F_D(p)$, descending price sorting is better than ascending price sorting for searchers; (2) given the price distribution $F_A(p)$, descending price sorting is better than ascending price sorting.

Now we prove statement (1). Given the price distribution $F_D(p)$, let CS_{DD}^1 and CS_{DA}^1 be searcher's total surplus under descending price sorting and ascending price sorting, respectively; let CS_D^0 be the total surplus for shoppers.[21] And let N_{DD} and N_{DA} be the expected number of

[21] Shoppers do not care the type of price sorting since they always buy from the lowest-priced high-type firm.

searches under descending price sorting and ascending price sorting, respectively. Likewise, define CS^1_{AD}, CS^1_{AA}, CS^0_A, N_{AD} and N_{AA} as the counterparts when the given price distribution is $F_A(p)$. To understand the notations, for the superscripts, 1 stands for "searcher" and 0 for "shopper"; for the subscripts, the first letter $S \in \{A, D\}$ stands for the given price distribution $F_S(p)$, and the second letter $S' \in \{A, D\}$ stands for the type of price sorting chosen by searchers.

Since searchers who use descending price sorting make the same purchases as shoppers do, we have

$$\begin{aligned} CS^1_{DA} &= CS^0_D - cN_{DA} \\ CS^1_{AA} &= CS^0_A - cN_{AA}. \end{aligned}$$

Moreover, we have

$$\lim_{c\to 0} N_{DA} = \lim_{c\to 0} N_{AA} = N(1-r) - (1-r)^N,$$

which implies that

$$\lim_{c\to 0} \frac{CS^1_{DA} - CS^1_{AA}}{c} = \lim_{c\to 0} \frac{CS^0_D - CS^0_A}{c} = 0,$$

according to the proof of Proposition 10.

Thus,

$$\begin{aligned} \lim_{c\to 0} \frac{CS^1_{DD} - CS^1_{DA}}{c} &= \lim_{c\to 0} \frac{CS^1_{DD} - CS^1_{AA}}{c} \\ &> 0, \end{aligned}$$

due to Proposition 10, which proves statement (1).

Finally, we prove statement (2). We take the price distribution $F_A(p)$ as given. According to the previous analyses, the searcher's optimal stopping rule under descending price sorting can be characterized by a group of reservation prices: $\hat{p}_1 \leq \hat{p}_2 \leq ... \leq \hat{p}_{N-1}$, where $\hat{p}_n$ solves

$$\int_{\underline{p}_A}^{\hat{p}_n} (\hat{p}_n - x) d\left(\frac{1-r+rF_A(x)}{1-r+rF_A(\hat{p}_n)}\right)^{N-n} = c, \text{ for } n = 1, ..., N-1.$$

Especially, the reservation price at stage $N-1$, $\hat{p}_{N-1}$, satisfies that

$$\int_{\underline{p}_A}^{\hat{p}_{N-1}} (\hat{p}_{N-1} - x) d\left(\frac{1-r+rF_A(x)}{1-r+rF_A(\hat{p}_{N-1})}\right) = c,$$

or

$$r\int_{\underline{p}_A}^{\hat{p}_{N-1}} F_A(x)dx = c\left[1-r+rF_A(\hat{p}_{N-1})\right]. \tag{45}$$

By taking the above optimal stopping rule, the searcher's expected surplus under descending price sorting is CS^1_{AD}.

Now consider the following suboptimal searching strategy under descending price sorting: at any stage, a searcher continues sampling if and only if the current price is higher than $\hat{p}_{N-1}$, which is given in (45). In other words, searchers behave as if the reservation prices at all stages are the same and equal to $\hat{p}_{N-1}$. Let $C\hat{S}^1_{AD}$ and $\hat{N}_{AD}$ be the searcher's expected surplus and the expected number of searches under this stopping rule, respectively. The suboptimality of the search strategy implies that $C\hat{S}^1_{AD} \leq CS^1_{AD}$. Thus, to prove statement (2), it is sufficient to show that $C\hat{S}^1_{AD} > CS^1_{AA}$ as long as the search cost is small, or

$$\lim_{c \to 0} \frac{C\hat{S}^1_{AD} - CS^1_{AA}}{c} > 0.$$

Similar to the proof of Proposition 6, one can easily see that

$$\begin{aligned} \lim_{c\to 0} \hat{N}_{AD} &= \lim_{c\to 0} N_{AD} = Nr - r^N \\ &< N(1-r) - (1-r)^N \\ &= \lim_{c\to 0} N_{AA}, \text{ when } r < 1/2. \end{aligned}$$

Under the above suboptimal stopping rule, we have that

$$\begin{aligned} C\hat{S}^1_{AD} &= \sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m \{ \int_{\hat{p}_{N-1}}^{a} (a-x)\,dG_m(x) + \\ & \sum_{s=1}^{m} C(m,s)\,(1-F_A(\hat{p}_{N-1}))^{m-s} \int_{\underline{p}_A}^{\hat{p}_{N-1}} (a-x)\,dH_s(x) \} \\ & -c\hat{N}_{AD}, \end{aligned}$$

where $G_n(x) = 1 - [1 - F_A(x)]^n$ and $H_n(x) = [F_A(x)]^n$, representing the distribution functions for the smallest order statistic and the largest order statistic for a sample of size n, respectively.

It is easy to express CS^1_{AA} as

$$CS^1_{AA} = \sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m \int_{\underline{p}_A}^{a} (a-x)\,dG_m(x) - cN_{AA}.$$

Thus, we have that

$$\begin{aligned} \lim_{c\to 0} \frac{C\hat{S}^1_{AD} - CS^1_{AA}}{c} &= \Delta + \lim_{c\to 0} \left(N_{AA} - \hat{N}_{AD} \right) \\ &> \Delta, \text{ when } r < 1/2, \end{aligned}$$

where

$$\begin{aligned}
\Delta &= \frac{1}{c}\sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m\{\int_{\hat{p}_{N-1}}^{a}(a-x)\,dG_m(x) \\
&\quad + \sum_{s=1}^{m} C(m,s)\,[1-F_A(\hat{p}_{N-1})]^{m-s}\int_{\underline{p}_A}^{\hat{p}_{N-1}}(a-x)\,dH_s(x)\} \\
&\quad - \frac{1}{c}\sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m\int_{\underline{p}_A}^{a}(a-x)\,dG_m(x).
\end{aligned}$$

It is then sufficient to prove that $\Delta = 0$. To see it, note that

$$\begin{aligned}
\Delta &= \lim_{c\to 0}\frac{1}{c}\sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m\{-\int_{\underline{p}_A}^{\hat{p}_{N-1}}(a-x)\,dG_m(x) \\
&\quad + \sum_{s=1}^{m} C(m,s)\,(1-F_A(\hat{p}_{N-1}))^{m-s}\int_{\underline{p}_A}^{\hat{p}_{N-1}}(a-x)\,dH_s(x)\} \\
&= \lim_{c\to 0}\frac{1}{c}\sum_{m=1}^{N} C(N,m)\,(1-r)^{N-m}\,r^m\{-\int_{\underline{p}_A}^{\hat{p}_{N-1}}G_m(x)dx \\
&\quad + \sum_{s=1}^{m} C(m,s)\,(1-F_A(\hat{p}_{N-1}))^{m-s}\int_{\underline{p}_A}^{\hat{p}_{N-1}}H_s(x)dx\} \\
&= \lim_{c\to 0}\frac{1}{c}\int_{\underline{p}_A}^{\hat{p}_{N-1}}\left\{\begin{array}{c}[1-rF_A(\hat{p}_{N-1})+rF_A(x)]^N-1 \\ -[1-rF_A(\hat{p}_{N-1})]^N+(1-rF_A(x))^N\end{array}\right\}dx \\
&= \lim_{c\to 0}\frac{\partial\int_{\underline{p}_A}^{\hat{p}_{N-1}}\left\{\begin{array}{c}[1-rF_A(\hat{p}_{N-1})+rF_A(x)]^N-1 \\ -[1-rF_A(\hat{p}_{N-1})]^N+(1-rF_A(x))^N\end{array}\right\}dx}{\partial c} \\
&= \lim_{c\to 0}\int_{\underline{p}_A}^{\hat{p}_{N-1}}NrF'(\hat{p}_{N-1})\frac{\partial\hat{p}_{N-1}}{\partial c}\left\{\begin{array}{c}-[1-rF_A(\hat{p}_{N-1})+rF_A(x)]^{N-1} \\ +[1-rF_A(\hat{p}_{N-1})]^{N-1}\end{array}\right\}dx \\
&= -\lim_{c\to 0}NrF'(\hat{p}_{N-1})\frac{\partial\hat{p}_{N-1}}{\partial c}\int_{\underline{p}_A}^{\hat{p}_{N-1}}\left\{\begin{array}{c}[1-rF_A(\hat{p}_{N-1})+rF_A(x)]^{N-1} \\ -[1-rF_A(\hat{p}_{N-1})]^{N-1}\end{array}\right\}dx \\
&= 0,
\end{aligned}$$

where in the last step we have used two facts

$$\lim_{c\to 0}F_A(\hat{p}_{N-1})\frac{\partial\hat{p}_{N-1}}{\partial c} = \frac{1-r}{r},$$ [22]

[22] See Equation (45).

and[23]

$$\lim_{c\to 0}\frac{1}{F_A(\hat{p}_{N-1})}\int_{\underline{p}_A}^{\hat{p}_{N-1}}\left\{\begin{array}{c}[1-rF_A(\hat{p}_{N-1})+rF_A(x)]^{N-1}\\ -[1-rF_A(\hat{p}_{N-1})]^{N-1}\end{array}\right\}dx=0.$$

■

Now we derive a high-type firm's demand function under descending price sorting, when M products can be observed in each sample. Let the pre-given reservation prices be $\hat{p}^M_{D,1}\leq\hat{p}^M_{D,2}\leq ...\leq\hat{p}^M_{D,\frac{N}{M}-1}$ and the conditional optimal price distribution for high-type firms be $F_D(p;\hat{p}_D,M)$.

Similar to the case in which $M=1$, define

$$\hat{p}^M_{D,0}\equiv\underline{p}^M_D\text{ and }\hat{p}^M_{D,\frac{N}{M}}\equiv\overline{p}^M_D=a.$$

We derive the high-type firm's demand function for each price interval, $p\in(\hat{p}^M_{D,n},\hat{p}^M_{D,n+1}]$, where $n=0,1,...,\frac{N}{M}-1$. let $D^M_0(p)$ and $D^M_1(p)$ be the demand from shoppers and searchers, respectively. Thus, the total demand is given by

$$D^M(p)=\mu D^M_0(p)+(1-\mu)\,D^M_1(p), \tag{46}$$

where $D_0(p)=[1-rF_D(p;\hat{p}_D,M)]^{N-1}$ since shoppers buy at price p if and only if all other products are either low-quality or have a price higher than p.

The demand from searchers is given by

$$\begin{aligned}D^M_1(p) &= \Pr\{\text{searchers buy at price } p\}\\ &= \sum_{k=0}^{N-1}\Pr\{\text{searchers buy at price } p\text{, with } k \text{ prices higher than } p\}\\ &= \sum_{k=0}^{N-1}D^M_1(p,k),\end{aligned} \tag{47}$$

where $D^M_1(p,k)=\Pr\{\text{searchers buy at price } p\text{, and there are } k \text{ prices higher than } p\}$, for $k=0,1,...,N-1$.

[23] Note that as $c\to 0$, we have $\hat{p}_{N-1}\to\underline{p}_A$ and $F_A(\hat{p}_{N-1})\to 0$, so that

$$\begin{aligned}0 &\leq \frac{1}{F_A(\hat{p}_{N-1})}\int_{\underline{p}_A}^{\hat{p}_{N-1}}\left\{\begin{array}{c}[1-rF_A(\hat{p}_{N-1})+rF_A(x)]^{N-1}\\ -[1-rF_A(\hat{p}_{N-1})]^{N-1}\end{array}\right\}dx\\ &< \frac{1}{F_A(\hat{p}_{N-1})}\int_{\underline{p}_A}^{\hat{p}_{N-1}}\left\{1-[1-rF_A(\hat{p}_{N-1})]^{N-1}\right\}dx\\ &= \frac{1-[1-rF_A(\hat{p}_{N-1})]^{N-1}}{F_A(\hat{p}_{N-1})}\left(\hat{p}_{N-1}-\underline{p}_A\right)\to 0.\end{aligned}$$

Denote $\hat{F}^M_{D,k} = F_D(\hat{p}^M_{D,k}; \hat{p}_D, M)$ for any $k = 0, 1, ..., \frac{N}{M} - 1$, with $\hat{F}^M_{D,0} = 0$. The searcher's demand is given as follows.

For any $p \in (\hat{p}^M_{D,n}, \hat{p}^M_{D,n+1}]$,
(i) if $k \leq M(n+1) - 2$, we have

$$D^M_1(p,k) = C(N-1,k)\,(1-r)^{N-1-k}\,[r - rF_D(p;\hat{p}_D,M)]^k\,,$$

(ii) if $k = M(n+1) - 1$, we have

$$D^M_1(p,k) = C(N-1,k)\,[1-r+rF_D(p;\hat{p}_D,M)]^{N-1-k}\,[r - rF_D(p;\hat{p}_D,M)]^k\,,$$

(iii) if $k = M(n+s) + i$, where $s = 1, ..., \frac{N}{M} - n - 1$ and $i = 0, ..., M-2$, we have

$$\begin{aligned} D^M_1(p,k) \quad &= \quad C(N-1,i)C(N-1-i, M(n+s))\,(1-r)^{N-1-i-M(n+s)} \\ &\quad \left(r - r\hat{F}^M_{D,n+s}\right)^{M(n+s)} [r - rF_D(p;\hat{p}_D,M)]^i\,, \end{aligned}$$

(iv) if $k = M(n+s) + M - 1$, where $s = 1, ..., \frac{N}{M} - n - 1$, then

$$\begin{aligned} D^M_1(p,k) \quad &= \quad C(N-1,M-1)C(N-M, M(n+s))\,[1-r+rF_D(p;\hat{p}_D,M)]^{N-1-k} \\ &\quad \left(r - r\hat{F}^M_{D,n+s}\right)^{M(n+s)} [r - rF_D(p;\hat{p}_D,M)]^{M-1}\,. \end{aligned}$$

To understand (i), note that if there are $k \leq M(n+1) - 2$ prices higher than p, price p will appear in one of the first n pages/samples. Since $p > \hat{p}^M_{D,n}$, searchers buy at price p only if all the products with prices lower than p are low-quality. Thus, in this case, $D^M_1(p,k)$ equals the probability that k out of $N-1$ prices are higher than p, and the rest $N-1-k$ prices are zero (i.e., they are low-quality products).

For expression (ii), when $k = M(n+1) - 1$, price p appears as the last one (or the lowest one) in page $n+1$. Since $\hat{p}^M_{D,n} < p \leq \hat{p}^M_{D,n+1}$, searchers never stop searching in the first n pages, and buy at price p immediately without further searches. Thus, $D^M_1(p,k)$ equals the probability that k out of $N-1$ prices are higher than p, and the rest $N-1-k$ prices are lower than p.

For expression (iii), if $k = M(n+s) + i$, where $s = 1, ..., \frac{N}{M} - n - 1$ and $i = 0, ..., M-2$, then price p is in the middle of page $n+s+1$ (not the last one). In this case, searchers purchase at price p if and only if two conditions are satisfied: (1) the first $M(n+s)$ prices should be no lower than the reservation price $\hat{p}^M_{D,n+s}$, so that searchers will not stop and purchase before they observe p in page $n+s+1$; (2) all the prices after p should be zero (i.e., low-quality products) because otherwise searchers will buy at a lower price in page $n+s+1$. Thus, $D_1^M(p,k)$ equals the probability that $M(n+s)$ out of $N-1$ prices are higher than $\hat{p}_{D,n+s}^M$, i prices are higher than p, and the rest $N-1-k$ prices are zero.

Finally, for expression (iv), when $k = M(n+s)+M-1$, where $s = 1, ..., \frac{N}{M}-n-1$, price p appears as the last one in page $n+s+1$. Searchers purchase at price p if and only if they did not stop searching in the first $n+s$ pages. In other words, the first $M(n+s)$ prices should be no lower than the reservation price $\hat{p}_{D,n+s}^M$. Compared to case (iii), that searchers purchase at price p does not require that all the prices after p be zero. This is because p is the last and the lowest price in page $n+s+1$. Thus, in this case, $D_1^M(p,k)$ equals the probability that $M(n+s)$ out of $N-1$ prices are higher than $\hat{p}_{D,n+s}^M$, $M-1$ prices are higher than p, and the rest $N-1-k$ prices are lower than p.

With the expression of $D_1^M(p,k)$ for any $k = 0, ..., N-1$, the total demand function is then derived according to (47) and (46). And the conditional optimal price distribution $F_D(p;\hat{p}_D, M)$ can be solved according to the constant-profit condition $\pi_D = pD^M(p)$, for any p within the price support.

Proof of Proposition 12. Let N_S^M be the expected number of searches that take place under price sorting $S \in \{R, A, D\}$, when M products are observed in each sample. Then it suffices to show that, as the search cost $c \to 0$,

$$\begin{aligned} N_A^M &< N_D^M < N_R^M, \text{ if } r > 1/2; \\ \text{and } N_D^M &< N_A^M < N_R^M, \text{ if } r < 1/2. \end{aligned}$$

Following the similar arguments as those in Proposition 6, we can see that, as the search cost approaches zero, under random price sorting, searchers always sample all the products in the market; under ascending price sorting, searchers stop sampling after they have gone through all the low-quality products; under descending price sorting, searchers stop sampling only when they have gone through all the high-quality products. Thus, N_A^M and N_D^M are always smaller than N_R^M. Moreover, $N_A^M > N_D^M$ if and only if there are more low-quality products in the market, i.e. $r < 1/2$. ■

Proof of Proposition 13. The logic is the same as that of Proposition 11. Random price sorting is never chosen in equilibrium because it is always dominated by ascending price sorting: compared to random price sorting, by choosing ascending price sorting, searchers can purchase high-quality products at lower prices and search less frequently. To compare ascending price sorting and descending price sorting, we can show that, given any price distribution, the difference between searchers' *purchase surplus* is an infinitesimal of higher order than the search cost c, as $c \to 0$. On the other hand, the total expected number of searches is smaller under ascending price sorting than under descending price sorting if and only if $r > 1/2$. This means there is always a unique equilibrium in the case of endogenous price sorting, in which consumers choose ascending price sorting if $r > 1/2$, and choose descending price sorting if $r < 1/2$. ■

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