Aggregation and multilevel design for systems: finding guidelines

Donald G. Saari
Professor: Mathematics, Economics
Director: Institute for Mathematical Behavioral Sciences
University of California
Irvine, CA 92697-5100
Email: dsaari@uci.edu

All areas of engineering have a need to find appropriate aggregated outcomes for systems. Issues range from decision problems, "divide-and-conquer" approaches that include aspects of multidisciplinary design optimization (MDO) and the effects of a division of labor for, perhaps, a design project, the inefficiencies that can accompany multidisciplinary projects involving, say, design, manufacturing, and sales, to the complexities of multiscale design, analysis, and even nanotechnology. But as shown, if the adopted approach (e.g., management choices, divide-and-conquer methodology, modeling of the biology/physics, decision rule, etc.) satisfies particular accepted practices, then certain complexities and inefficiencies must be anticipated. A disturbing corollary is that even should "success" appear to have been achieved with an approach that satisfies these conditions, it need not be as firm as expected. Ways to improve methodologies must avoid the specified conditions.

1 Introduction

Multiscale design and analysis are interdisciplinary developments with a promise to address inherent system complexities that occur in engineering and elsewhere. Accompanying the objective of relating behavior from different scales, and even from different disciplines (e.g., MDO), is the expectation that a combined micro-macro analysis may be more tractable and useful than what can be achieved by just examining micro level behavior in detail (Weinan et. al. [1]). As true in nanotechnology, for instance, information about the macro structure may help to identify which variables and aspects of micro and nano effects to emphasize.

While multiscale and multidisciplinary methodologies promise advantages, particularly when coupled with computational methods, its exploratory nature is reflected with cautionary word choices such as expressing a "hope" of finding connections (e.g. [1]). This caution reflects both the inherent complexity of these problems and a recognition that many of the approaches are approximation techniques[1] developed to manage the complexity.

If approaches cannot provide optimal answers for analyzed and/or designed settings, then methodological improvements are needed. To find improvements, a crucial first step is to understand what can go wrong. That is, a goal should be to discover what hinders our attempts to find reasonable (leave alone optimal) outcomes and to obtain information about which aspects of an approach should be changed. This defines the objective of this article; it is to identify a common source of difficulties that afflicts various multiscale and multidisciplinary approaches for design and/or analysis. As these results address complexities that arise with a variety multiscale and multidisciplinary approaches, this broader topic is called "multilevel methodology."

A way to analyze multilevel methodologies is to treat them as generalizations of aggregation processes. Doing so suggests where to find insights from other literatures. In return, this transferability of conclusions allows the conclusions developed here to be applied to issues as varied as the economic "supply and demand" story, questions from astronomy, organizational design, and "parts-whole" ecological effects.

For a flavor of what will be described, consider a common approach used to examine a complex system which is to first decompose the problem into parts and then analyze each component. Accompanying this divide-and-conquer procedure is a "path dependency" difficulty. This is where even if reasonable outcomes are found for each part, the final outcome could more accurately reflect the order in which the parts are analyzed and/or assembled rather than the actual structure. To illustrate this phenomenon with a decision rule where the goal is to find a reasonable choice (macro outcome) from among seven alternatives, suppose the rankings of these alternatives over fifteen criteria (the micro behavior)

[1] A point that was strongly reinforced by a referee's comments.

are

$$
\begin{array}{l}
5\ A \succ B \succ C \succ D \succ E \succ F \succ G, \\
5\ B \succ C \succ D \succ E \succ F \succ G \succ A, \\
5\ C \succ D \succ E \succ F \succ G \succ A \succ B
\end{array} \quad (1)
$$

Each criterion ranks $C \succ D \succ E \succ F$ over G, so G clearly is the inferior choice.

The comprehensive Eq. 1 information typically is not available, so a common decision technique is to compare the alternatives in pairs where, at each step, the poorer one is dismissed. But with the Eq. 1 data, rather than this paired comparison method ensuring a reasonable choice, *each* alternative—even G—could be identified as a reasonable or even the "best" outcome just by using different orders of comparisons. To illustrate with a path that selects G, compare the $\{E, F\}$ winner (which is E) with D; compare that winner (D) with C; that winner (C) with B; that winner (B) with A; and that winner (A) with G to reach the final outcome of G. As each paired outcome is determined with a two-thirds or unanimous support, it might appear that strong evidence supports and validates this (faulty) conclusion of G. A message derived from this example is that while an approach may deliver reasonable answers in some settings, when the same technique is used in other settings, the path dependency phenomenon can provide seemingly "strong evidence" to support an inferior, inefficient, or even incorrect conclusion.

The "path dependency" phenomenon is so common that it must be anticipated. Already in a calculus course that introduces the notion of path integrals, students learn that the work required to go from one point to another can depend upon the path taken. As shown here, this "path dependency" phenomena extends to *all* aggregation areas: Expect it to affect multilevel methodologies. In particular, anticipate path dependency to adversely affect the outcomes for location problems, assessments, engineering and manufacturing decisions, issues of organizational design, nanotechnology, multidisciplinary design optimization, and even the common "division of labor" practice of assigning tasks to different groups to achieve a desired outcome.

Because my goal is to understand what causes difficulties with multilevel methodologies, my main conclusions (Sect. 3) are necessarily negative: they assert that whenever a procedure (e.g., an approximation technique, decision rule, or any other multilevel approach) satisfies certain natural and convenient properties, the methodology need not, in general, produce appropriate, or even reasonable conclusions about the system; i.e., the developed answers about the system may be misleading and even incorrect. An associated issue (which is an ongoing project) is to find ways to circumvent these negative consequences. Resolutions will be issue-specific, but intuition about what needs to be done (Sect. 5) comes from Rubik's Cube.

2 Arrow's result

My results are motivated by Arrow's [2] profound but negative theorem, which asserts that it is impossible to create a decision rule that always does what seems to be obviously possible to do. Arrow's result is based on the ordinal rankings of a finite number of alternatives. Similar to Arrow's assertion, my results prove that much of what we try to do cannot be done. In contrast with Arrow's theorem, my conclusions address a much broader set of issues than allowed by his setting of ordinal rankings of a finite number of alternatives. By being related to Arrow's seminal result, my assertions contribute to the growing literature that explores the implications of Arrow's theorem for engineering.

This literature appears to have started when Hazzelrigg [3] called attention to the close connection between social choice and engineering decisions. Subsequent papers, such as Franssen [4], van de Poel [5], and Wassenaar and Chen [6] expand on this connection while others, such as Scott and Antonsson [7], caution that Arrow's theorem is not fully applicable because the "foundation of many engineering decision methods is the explicit comparison of degrees of preference, a comparison that is not available in the social choice problem." While they are correct, the more general results developed here fill this gap because these conclusions go far beyond the ordinal rankings required by Arrow to include comparisons that do involve a continuum of choices, degrees of preference, and even interactive and dynamical properties. To indicate what must be done to find positive assertions, a new interpretation of Arrow's result is developed at the end of this section and discussed in Sect. 5. An advantage of this new explanation is that it indicates why approaches that are used to understand complexities from engineering or physics can encounter serious problems.

The connections between multilevel methodologies and Arrow's Theorem make it appropriate to start with Arrow's result [2]. (For a significantly different, benign interpretation of Arrow's seminal theorem, which leads to positive conclusions, see Saari [8, 9].) Arrow imposes a compatibility constraint on the voters' admissible inputs: individual rankings must be "orderly" in that they are complete (each voter ranks each pair of alternatives) and transitive; i.e., if a voter prefers $A \succ B$ (i.e., A is preferred to B) and $B \succ C$, the voter must prefer $A \succ C$. To interpret Arrow's result in a decision context, replace "voters" with "criteria."

Assumption 1. *(Compatibility) Each voter ranks the $n \geq 3$ alternatives in a complete, transitive manner. There are no restrictions on a voter's choice of a ranking.*

The next assumptions describe seemingly reasonable conditions that any useful decision rule might be expected to satisfy. The first, Pareto, requires the rule to respect unanimity.[2] To motivate the second, Independence of Irrelevant Alternatives (IIA), suppose a panel evaluating research proposals finds that Alice $\succ$ Barb. IIA requires this Alice $\succ$

[2] The Pareto condition can be replaced with the much weaker condition where at least each of two pairs of alternatives have at least two different societal outcomes (Saari [8]).

Barb conclusion to hold independent of how the panel views Connie's proposal.

Assumption 2. *(Pareto) If all voters rank a pair of alternatives in the same way, then this common ranking is the pair's societal ranking.*

Assumption 3. *(Independence of Irrelevant Alternatives, IIA) The societal ranking for any pair of alternatives, say $\{A,B\}$, is based strictly on how the voters rank this pair. Namely, for any two profiles (a profile specifies how each voter, or criterion, ranks the alternatives) $\mathbf{p}_1$ and $\mathbf{p}_2$ in which each voter has the same $\{A,B\}$ ranking, both profiles have the same $\{A,B\}$ societal ranking.*

The final assumption imposes structure on the societal outcome; it merely requires the societal ranking of the alternatives to be transitive.

Assumption 4. *Societal outcomes are complete, transitive rankings.*

It is reasonable to expect that most group decision rules satisfy these innocuous appearing conditions, but Arrow's Theorem asserts that only one rule does.

Theorem 1. *(Arrow) With $n \geq 3$ alternatives and at least two voters, the only rule that always satisfies Assumptions 1-4 is a dictator. Namely, the rule is identified with a specific individual in that, for all possible profiles, the societal outcome always coincides with that voter's ranking.*

As criteria replace voters in decision problems, the conclusion is that if a decision rule always satisfies these conditions, then its outcomes are strictly based on information coming from a single criterion; i.e., the decision rule always ranks the alternatives as ranked by the specified criterion.

The impact of Arrow's result derives from the reality that we do not use dictatorial voting rules; we do not use decision rules that depend solely on a single criterion. As such, Thm. 1 guarantees that whatever rule is adopted, even should there be circumstances where the rule satisfies all of the specified requirements, other situations must exist where at least one of the conditions is violated. To find which provision can breached, first determine which requirements the rule must always satisfy; settings exist where a remaining condition cannot be always observed.

In engineering comparisons, for instance, it is not uncommon to use versions of majority (or stronger) votes over pairs. This non-dictatorial rule clearly satisfies Pareto and IIA. The only remaining condition is the transitivity of outcomes, so, according to Thm. 1, there *must* exist data sets of transitive rankings where the outcome is not transitive. An example, which is central to Hazzelrigg's arguments [3], is exhibited by the Condorcet triplet:

Preferences	$\{A,B\}$	$\{B,C\}$	$\{A,C\}$
$A \succ B \succ C$	$A \succ B$	$B \succ C$	$A \succ C$
$B \succ C \succ A$	$B \succ A$	$B \succ C$	$C \succ A$
$C \succ A \succ B$	$A \succ B$	$C \succ B$	$C \succ A$
Outcome	$A \succ B$	$B \succ C$	$C \succ A$

(2)

The cyclic ranking of the bottom row, where each victory is supported by the same 2:1 count, frustrates the decision process; e.g., although A beats B, and B beats C, A need not be a reasonable choice because C beats A. This cyclic effect is what creates the inefficiencies of the "path dependency" problem (which is one of several consequences of Arrow's result). A realistic concern is that when this difficulty arises in practice, it probably would not be recognized.

The following new interpretation of Arrow's theorem is one that more closely addresses concerns related to the analysis of complexity issues from engineering. The goal is to find the appropriate ranking (macro level) of the alternatives for a given profile; Arrow's theorem asserts that there are settings where this objective must fail with a rule that satisfies IIA and Pareto! Notice; IIA and Pareto require the decision rule to construct a system's outcome in terms of paired comparisons. Arrow's result, then, guarantees that these divide-and-conquer approaches must experience serious difficulties, which include the path dependency problem. Thus

> Arrow's Theorem asserts that when a rule uses the inputs from two or more agents (or criteria), and it is based on IIA and the Pareto conditions, then there always exist situations where the true structure of the whole need not resemble the answers developed by using information from the assembly of the parts.

In other words, if such a divide-and-conquer approach is used to analyze the complexity of a system, it is guaranteed to encounter difficulties. One must wonder whether related assertions apply to "divide-and-conquer" approaches of the kind that are commonly used in multilevel methodologies. They do; identifying one such extension is the purpose of this paper.

3 A generalized inconsistency theorem

Problems without solutions are seldom published, so the literature provides limited information about which methodologies are associated with the difficulties of multilevel system approaches (whether for engineering, physics, biology, design, mathematics, or organizational structures). An alternative, effective way to discover where these difficulties occur is to question experts about what they do. A standard response involves various "divide-and-conquer" approaches where the system is decomposed into component parts. Similar to what is done with Eq. 1, the next step is to determine connections and appropriate outcomes among scales or levels for each component of the system. This may be accomplished, for instance, by using particular laws of physics, biology, or the expertise of experts. The final step, which is to assemble the answers developed for the component parts into a compatible outcome for the system, often requires adjustments, guesswork, and "muddling." Part of my analysis is to explain why this last step can be difficult and frustrate achieving success.

My main result alerts users about subtle limitations that accompany this general methodology; it identifies what causes basic problems. An importance of this conclusion is that unless we know the source of problems, we run the danger of continually repeating them. Knowing what causes the limitations, on the other hand, may suggest how to modify these natural divide-and-conquer approaches to create a related methodology that is at least a step or two more efficient.

The spirit of my conclusions is revealed by the inclusion of Sects. 1 and 2. For instance, while a divide-and-conquer methodology might enjoy success, it will be shown that other settings exist where either it is impossible to piece together information about the parts to obtain conclusions, or inappropriate, inefficient outcomes arise. What exacerbates the problem is that it can appear that the faulty conclusions are supported by strong evidence; i.e., expect the Eq. 1 path dependency problem to accompany these techniques. Moreover, should such a methodology permit successful conclusions in a setting, in other settings where the assured inappropriate outcomes arise, a natural attempt to resolve difficulties might be to emphasize the particulars of the specific project rather than to address the true source of the problem—methodological limitations.

Not all divide-and-conquer approaches experience these problems, but many do. To identify when a methodology should be questioned, I model a class of approaches where the way in which a system is divided into its component parts, and answers are sought, ensures the existence of difficulties. While my result captures all settings from different disciplines that were described to me (e.g., where it proved difficult to achieve reasonable outcomes), it is not complete; my result does not identify all approaches that can be afflicted by these complications. So, if a multilevel methodology even resembles this modeling, caution should be exercised about interpreting and trusting conclusions.

My modeling reflects a multilevel feature (to reflect concerns raised by colleagues in nanotechnology) that adequate information about what happens at certain levels may not be known; e.g., in multiscale design, rules of interaction at the nano scale need not be fully understood. Indeed, an explicit objective of the analysis may be to obtain insight into what happens at a micro or nano level in terms of what is known about the macro level. To capture this intent, the structure of a system at a macro level is modeled in a way to include where macro information can be used to identify potential structures at micro and other levels. This modeling also includes, of course, settings where the micro structures are understood.

Macro level: Assume that the system at the macro level, or the space of "aggregated parts," is divided into three or more component sections $\{\mathcal{C}^j\}$: Treat these system components as being independent of each other (in the weak sense described below). The actual choices of $\{\mathcal{C}^j\}$ components depend on what is being analyzed; they describe how a practitioner decides to decompose a multilevel system into parts. It is reasonable, for instance, to divide an industrial setting into the design component $\mathcal{D}$, manufacturing component $\mathcal{M}$, and sales component $\mathcal{S}$ that consist, respectively, of all ways to design, manufacture, and market a product. Similarly, a design project might be divided among three or more units: all available options for each unit $\mathcal{D}^j$ constitute a component. In a multiscale problem, each component could consist of the aggregated effects of what happens at a micro level. With health policy, the macro effects may consist of the spread of different diseases, which reflect the aggregated way in which, at the micro level, the dynamics of how healthy and infected individuals interact with one another. In analyzing the structure of a galaxy (or group of galaxies), the major components include the total mass, rotational velocity, and luminosity (e.g., Zwicky [10]).

In many settings, such as $\mathcal{D} \times \mathcal{M} \times \mathcal{S}$, "independence" is ensured with $\mathcal{C}^i \cap \mathcal{C}^j = \emptyset$ for pairs; each $\mathcal{C}^j$ component consists of different entities. In other settings, the purpose of each $\mathcal{C}^j$ could be to extract different information from a common set. This occurs in Arrow's setting where, for three alternatives $\{A, B, C\}$, the $\mathcal{C}^j$ set capturing $\{A, B\}$ rankings has two elements. The "$A \succ B$" element is the *subset* $\{A \succ B \succ C, A \succ C \succ B, C \succ A \succ B\}$ and the cyclic ranking defined by $A \succ B$; the "$B \succ A$" element is the *subset* $\{B \succ A \succ C, B \succ C \succ A, C \succ B \succ A\}$ with the cyclic ranking defined by $B \succ A$. Thus each element is, respectively, the subset of all rankings where $A \succ B$, and the subset of rankings where $B \succ A$. Similarly, the $\{A, C\}$ set consists of all rankings where $A \succ C$ and all where $C \succ A$. Notice; each ranking in each $\{A, C\}$ element is a ranking in some $\{A, B\}$ element. But as the elements do not agree, they extract different information from the common set. All that is needed here is that no two $\mathcal{C}^j$, $\mathcal{C}^i$ sets, $j \neq i$, agree.

While I require each $\mathcal{C}^j$ component to have at least two elements (because there is nothing to analyze with a single element), it could include any number of alternatives, even a continuum of possibilities or degrees of comparison of the kind that can arise in engineering decisions. It may consist of the various configurations of molecules in a chemical setting, different design proposals for a project, or whatever is being modeled. For many issues, such as those from physics, elements of a component can include dynamics and even dynamical interactions. This holds for the galaxy example where rotational velocities form one of the components; it also includes any form of dynamic interactions. In other words, the actual contents of the components do not matter; my conclusions depend only on how they interact and how relationships among levels are determined.

Although the components are independent (as described above), the divide-and-conquer methodologies addressed here relate the component parts through *compatibility conditions*.[3] After all, it is easy to imagine a $(d, m, s) \in \mathcal{D} \times \mathcal{M} \times \mathcal{S}$ combination reflecting, respectively, design, manufacturing, and sales proposals that are not compatible; it is easy to imagine an incompatible $(d_1, d_2, d_3) \in \mathcal{D}^1 \times \mathcal{D}^2 \times \mathcal{D}^3$ consisting of the choices developed by three different units for a combined design project; it is easy to see how the parts of a health plan, or a foreign policy for a country, could be contradictory or counterproductive.

[3] If a divide-and-conquer method does not have compatibility requirements, it may be immune from the negative assertions developed here.

The "dark matter" concern from astronomy reflects the fact that the computed values of velocity, luminosity, and mass fail to be compatible with Newtonian theory (Zwicky [10]). The compatibility condition for Arrow's Theorem requires the binary rankings of pairs of alternatives to define a transitive ranking. The purpose of the compatibility condition used here (and given below) is to describe the structure of which combinations are, or are not, admissible.

Let me stress that the choice of the compatibility constraints are for you to select. Remember, the theme of this paper is to identify when the standard "divide-and-conquer" multilevel methodology encounters problems, so the compatibility conditions are based on the way in which a system is divided into parts; they describe how the parts are related. Thus these conditions can be used to identify what we want, and do not want, to occur; e.g., the compatibility condition could be defined to avoid combinations that cause inefficiencies or even failures.[4] The precise definition of "compatibility," then, depends on what is being examined, what is desired, and, in particular, how to reconstruct a system that is divided into component parts. For the astronomy concern of dark matter, the compatibility conditions require the mass, velocity, and luminosity values to be consistent with accepted physical laws.

To eliminate special cases and emphasize only $\mathcal{C}^k$ components that are meaningful in our search for problems that can occur with this methodology, treat a combination as being incomplete if it fails to include an element from each $\mathcal{C}^k$ component. The compatibility condition is defined to avoid redundant settings by including only relevant parts from each component $\mathcal{C}^k$. If, for instance, the compatibility conditions are determined by physical laws, engineering principles, or economic constraints, the condition eliminates all elements from each $\mathcal{C}^k$ that would never occur. With $\mathcal{D}\times\mathcal{M}\times\mathcal{S}$, for instance, it eliminates all manufacturing approaches m^* for which the resulting product always is impossible (or inefficient) to design and/or sell. Namely, if for all $d\in\mathcal{D}, s\in\mathcal{S}$, the combination (d,m^*,s) always fails the compatibility condition, then drop m^* from $\mathcal{M}$. To capture the above, I require each $c_k\in\mathcal{C}^k$ to be in at least one acceptable combination.

Similarly, to avoid being encumbered with "obvious" settings and to stress the relevancy of each $c_k\in\mathcal{C}^k$, the condition requires each c_k to be in some unacceptable combination.[5] To illustrate with the compatibility condition of transitive ranking of pairs, any specified ranking, say $A\succ B$ is in some transitive ranking, such as $A\succ C\succ B$, but it also is in intransitive settings such as where $B\succ C, C\succ A$. With the galaxy example, certain velocity values are acceptable when coupled with some mass and luminosity choices, but the same velocity choice can be unacceptable with other mass values. Using $\mathcal{D}\times\mathcal{M}\times\mathcal{S}$, this condition requires each design $d\in\mathcal{D}$ to be compatible with certain manufacturing and marketing proposals, but incompatible with other choices of manufacturing and/or sales proposals; in a physical setting combining angular momentum with, say, mass and total energy, some levels of angular momentum are acceptable with certain choices of mass and energy, but not with others.

To reflect the reality of organizational and engineering design, of physics, of economics, etc., the modeling of these compatibility conditions must allow some flexibility. There are several ways to do this, where each leads to similar conclusions. Motivation for the simple choice used here comes from physics where changing one of several forces can destroy an equilibrium setting, but it may be possible to compensate by making appropriate changes in a different force to return the system to an equilibrium status. The dark matter concern, for instance, searches for an appropriate amount of matter in a galaxy to return the variables to a compatible setting. As another illustration, envision a compatible combination (d,m,s) from $\mathcal{D}\times\mathcal{M}\times\mathcal{S}$, but where the marketing program $s'\in\mathcal{S}$ would be more profitable. Replacing the original s with s', however, may make the combination (d,m,s') incompatible; e.g., the way the product is manufactured may not match what is needed with sales. The compatibility conditions reflect what is done in practice; it is to make compensating adjustments in some other component; e.g., there may be a $m'\in\mathcal{M}$ that compensates for s' to make (d,m',s') compatible. With the $A\succ B, B\succ C, C\succ A$ setting, reversing any one ranking returns to the transitivity compatibility condition.

This flexibility condition is purposely designed to capture common ways that are used to adapt to changing circumstances. In a design project, while $(d_1,d_2,d_3)\in\mathcal{D}^1\times\mathcal{D}^2\times\mathcal{D}^3$ may be compatible, the material required to achieve d_1 may not be available, so it must be replaced with d_1^*. If the new (d_1^*,d_2,d_3) is incompatible, a natural approach is to determine whether it is possible to modify, say, d_3 to d_3^* so that the adapted combination (d_1^*,d_2,d_3^*) is compatible.

The above discussion is formalized with the three parts of the following definition. To distinguish combinations $(c_1,\dots,c_n)$ that are acceptable "outcomes", the notation is changed to $(o_1,\dots,o_n)$.

Definition 1. *A compatibility condition imposed on combinations $(c_1,\dots,c_n)\in\mathcal{C}^1\times\dots\times\mathcal{C}^n$ is "acceptable" if it satisfies the following three conditions:*

1. *(Completeness) An acceptable combination has a term from each $\mathcal{C}^k$.*
2. *(Meaningful) For each component $\mathcal{C}^k$, each $c_k\in\mathcal{C}^k$ is in combinations that satisfy the compatibility conditions and in combinations that do not. Namely, for each $c_k\in\mathcal{C}^k$, there exist a combination $(c_1^*,\dots,c_{k-1}^*,c_k,c_{k+1}^*,\dots c_n^*)$ that satisfies the compatibility condition. However, there is a compatible combination $(c_1',\dots,c_{k-1}',c_k',c_{k+1}',\dots c_n')$ where, by changing c_k' to c_k, the combination $(c_1',\dots,c_{k-1}',c_k,c_{k+1}',\dots c_n')$ fails the compatibility condition, $c_j^*,c_j'\in\mathcal{C}^j$.*

[4]As an illustration, in an ongoing project with A. Chandra, the compatibility conditions are, as of now, unspecified conditions about a certain process that can cause cracks in the product. Part of the project is to use the information from this paper to identify the appropriate compatibility conditions.

[5]This condition helps to capture my objective of understanding what causes incompatible outcomes; i.e., if a particular element always is in compatible outcomes, it probably does not cause the experienced difficulties, so ignore it. Anyway, engineering issues seldom are so accommodating.

3. *(Compensative) For any two components, j and k, there exists a combination $(c_1,\ldots,c_n)$ satisfying the compatibility condition where a change in the j^{th} component can be made to create an incompatible combination, but another change can then be made in the k^{th} component to make the new combination acceptable.*

A combination that satisfies these conditions is called an acceptable outcome; it is denoted by $\mathbf{O}=(o_1,\ldots,o_n)\in\mathcal{A}$.

Micro level: Rather than describing what happens among several levels, only a "micro level" is considered here. This is because a two-level discussion is easier to follow, and all results extend to multilevel settings.

The space of inputs, or the micro level parts, could mimic the macro level structure; e.g., with the $\mathcal{D}\times\mathcal{M}\times\mathcal{S}$ industrial example, the micro parts used to determine a final design may also come from $\mathcal{D}$, the sales proposals being examined may come from $\mathcal{S}$. In other settings, micro components can differ significantly from that of the macro level. This is true if there is incomplete information about the structure of the micro level, which often is true in nano design.

Whatever is done, assume that some scenario, a "best case one" if possible, identifies what micro parts, when combined in appropriate ways, determine a reasonable or appropriate element for each $\mathcal{C}^i$. Denote the identified collection of "parts" at the micro (or any other level) associated with the macro component $\mathcal{C}^j$ as $\mathcal{C}^{j*}$. In the industry example $\mathcal{D}$ represents the space of accepted designs, so $\mathcal{D}^*$ represents potential designs that are being considered at the planning stage. The nature (physical, biological, etc.) of the elements in $\mathcal{C}^j$ typically differ significantly from that of the $\mathcal{C}^{j*}$ elements; this is because $\mathcal{C}^j$ consists of macro effects while $\mathcal{C}^{j*}$ identifies micro effects.

Definition 2. *If the space of outcomes is given by $\mathcal{C}^1\times\ldots\times\mathcal{C}^n$, then the space of inputs is given by $\mathcal{C}^{1*}\times\ldots\times\mathcal{C}^{n*}$ where $\mathcal{C}^{j*}$ is the space of inputs associated with the outcomes in $\mathcal{C}^j$. "Acceptable combinations" of inputs satisfy the compatibility conditions of Def. 1. An acceptable combination is called a "plan." It is denoted by $\mathbf{p}=(c_1,\ldots,c_n)\in\mathcal{A}$.*

Again, the compensative condition may reflect equilibrium constraints coming from engineering, physics, and even economics. In the "supply equals demand" pure exchange model from economics, each unit (which defines a component) decides what to sell and buy according to a budget constraint: The money earned by selling some commodities equals the cost of buying other commodities. If a change is made in how much of a particular commodity is bought or sold, the budget constraint can be re-established with compensating changes in another commodity.

While the compatibility conditions for the inputs and outcomes may share the same characteristics, more often the macro and micro conditions differ significantly. (They must when the nature of macro and micro elements differ.) To illustrate with the "supply-demand" description, the compatibility conditions at the macro level are more severe; in addition to the budget constraint, for each commodity the sum of what is offered must equal what is requested.

Connections: In the informal description of the multilevel methodology, after dividing a system into parts, appropriate outcomes are found for each component. My modeling of this stage reflects how many multilevel projects link the micro and macro relationships.

This link, this aggregation rule, determines an outcome as based on information, or plans, that come from different "participants." In Arrow's setting the participants are voters, in a decision rule the participants are the different criteria, in a design project the participants can be the different units, in the $\mathcal{D}\times\mathcal{M}\times\mathcal{S}$ setting the three participants are the design, manufacturing, and sales units, in a physical behavior or multilevel analysis the participants could be competing physical/biological forces that contribute to the final effect, in the galaxy example the three participants are the computational ways that determine the total mass, velocity, and luminosity.

Assumption 5. *(Compatibility; inputs) Assume there are $m\geq 2$ participants. Each participant selects a plan $\mathbf{p}\in\mathcal{A}$; there are no restrictions on each participant's choice. The list of plans for the participants is called a profile; it is represented by $\mathbf{P}=(\mathbf{p}_1,\ldots,\mathbf{p}_m)$.*

Each participant must put forth a "plan," which is a proposal that is consistent across the board; e.g., the design participant must advance a design that is compatible with acceptable manufacturing and sales options. There are settings, of course, where a "participant's plan" ignores consistency conditions; e.g., the way in which rotational velocities in a galaxy are computed ignores luminosity effects. In these situations, assume that the consistency conditions are satisfied by de fault; the main theorem given below not only still holds, but it becomes easier to prove.

Assumption 5 suggests that each participant is subject to the same compatibility conditions, but this is not necessary; it is easy to envision settings, such as where the participants represent different physical/biological forces, where the contents of the components (one is physical and the other biological) and the compatibility conditions change with the participant. These situations remain subject to my conclusions because it is the *structure* of the compatibility conditions, rather than the actual choice of elements or specific conditions, that matters. For convenience of exposition everything is expressed as though all participants use the same constraints, but slight modifications extend everything to where the content and specific compatibility conditions can differ with the participant as long as they satisfy the Def. 1 *structures*.

The aggregation rule combines plans from the profile to create an outcome $\mathbf{O}$. How this is done is determined by the physics, biology, economics, organizational, or engineering principles. The next two assumptions capture basic properties of many multilevel rules. The first is that unanimity dominates when it occur. As an illustration, if the design, manufacturing, and sales plans are, respectively, (d_d,m^*,s_d), (d_m,m^*,s_m), and (d_s,m^*,s_s), then the universal agreement of m^* for the manufacturing component requires m^* to be the $\mathcal{M}$ entry for the outcome $\mathbf{O}$. This leads to the following Pareto condition.

Assumption 6. *(Pareto) If the space of inputs and outcomes have the same components, and if for a profile* $\mathbf{P}$*, there is some k where the* $\mathcal{C}^{k*}$ *entry in all* $\mathbf{p}_j$ *plans is the same* c_k*, then this agreed upon* c_k *term is the* $\mathcal{C}^k$ *entry in* $\mathbf{O}$.

If the output elements do not agree with that of the inputs, then identify every element $o^k \in \mathcal{C}^k$ *with a component* $c_k \in \mathcal{C}^{k*}$ *in that if* c_k *is the* $\mathcal{C}^{k*}$ *entry for each plan in a profile, then* $o^k \in \mathcal{C}^k$ *is the outcome;* $k = 1, \dots, n$.

The next assumption reflects a standard practice; it is where experts make the decision in their area of expertise. The assumption includes settings where the conclusion for a component is determined in terms of particular engineering/physical/biological principles. Using my standard $\mathcal{D} \times \mathcal{M} \times \mathcal{S}$ example, as long as the design unit strictly adheres to plans (so they are compatible), this unit is best qualified to determine the $\mathcal{D}$ component for the outcome. In other settings, the "expert participant" may represent a particular physical or biological force; e.g., a system's angular momentum is determined by certain physical principles.

Definition 3. *(Decentralization) A decisive participant is one that determines the outcome for a specified component for the outcome* $\mathbf{O}$.

Among the many examples that illustrate these conditions, a striking one comes from multidisciplinary design optimization (MDO) where "...in a multilevel approach, the overall analysis and optimization problem is decentralized into multiple interacting subproblems and the optimization is performed in each subproblem while they all work together in concern to obtain the solution to the MDO problem." (Li and Azarm [11].)

Main result: These minimal, seemingly innocuous assumptions are desirable and reflect what is done in practice, but they already cause a negative conclusion. (Theorem 2 more closely resembles Sen's Theorem [12] than Arrow's Theorem.)

Theorem 2. *With two or more decisive participants, no rule exists where the plans and outcomes always satisfy, respectively, compatibility assumptions 5 and Def. 1, and the rule satisfies the Pareto assumption 6.*

This theorem carries the disturbing message that, even should all participants strictly adhere to the compatibility conditions, the divide-and-conquer process of determining a system outcome by seeking "excellence" (or at least seeking a reasonable entry for each component) can cause inefficiencies and/or incompatible outcomes. Thus the troubling Sect. 2 kinds of problems also occur with those divide-and-conquer multilevel methodologies—including decision rules—that satisfy Thm. 2 conditions; i.e., if a divide-and-conquer rule satisfies the specified structural conditions, expect environments where either this approach *cannot* discover a conclusion, or the answer is wrong. If the Pareto condition does not apply, but if each component's outcome is determined by particular experts (e.g., specific physical/biological/engineering laws or principles), then the same assertion follows and the proof is essentially the same.

Stated in a different manner, even if appropriate system structures at different levels and a connection among them exists, there will be settings where this divide-and-conquer methodology cannot find the actual answer. In other words and illustrating with a setting where the "experts" are physical laws, it is not the physics that is at fault; it is the divide-and-conquer methodology.

Comments illustrating Thm. 2 also come from the MDO literature. The conclusion means it can be difficult to match results obtained about the "parts" into a general conclusion, so it closely matches the second part of the comment that "Decomposition-based optimization strategies are used to solve complex engineering design problems that might be otherwise unsolvable. Yet, the associated computational cost can be prohibitively high due to the often large number of separate optimizations needed for coordination of problem solutions." (Alyaquot, Papalambros, and Ulsoy [13].)

Because the structural conditions that cause these negative conclusions are specified, this theorem also suggests how to improve such methodologies. For a first cut, it requires restructuring how different levels (e.g., experts, reliance on physical laws, etc.) are connected. In particular, the way in which answers are determined for the component parts must involve forms of interactions that violate the Thm. 2 conditions; e.g., finding the appropriate choice for one component part of a system must more intimately depend upon and involve the search for the choices of other component parts.[6]

As with Arrow's theorem, Thm. 2 asserts that while settings might exist where everything appears to be satisfactory, caution is required because there must exist other settings where the results are incompatible, suffer inefficiencies, or are not reasonable. Because of the path dependency problem, even should "success" appears to be attained, it is not clear whether the outcome is a reasonable choice.

The modeling of the components goes beyond static settings to allow dynamics and adaptive approaches. If an initial design or manufacturing practice, for instance, has flaws, then adjustments are made to eliminate these difficulties. The need to make such an adjustment, of course, manifests an inefficiency that adds to the cost. Moreover Thm. 2 still applies; e.g., because of the path dependency issue, we may not know whether a "corrected" solution is reasonable or inferior.

While the result holds for any structure of the components, it is easier to illustrate with discrete examples. In this spirit, a simple example capturing the idea of the proof (Sect. 7) with $\mathcal{D} \times \mathcal{M} \times \mathcal{S}$ is where the only unacceptable outcomes are (d_1, m_1, s_1) and (d_2, m_2, s_2). Suppose there are two decisive participants where the first is decisive over $\mathcal{D}$ and the second over $\mathcal{S}$. The following table specifies each participant's plan where the dash indicates irrelevant information because the outcome is determined by a different decisive

[6] As an illustration of how these kinds of problems are avoided in a special case, an integrated way is developed in Saari [14] to relate the total mass and rotational velocities of a galaxy.

participant.

Participant	Plan	$\mathcal{D}^*$	$\mathcal{M}^*$	$\mathcal{S}^*$
1	(d_1, m_1, s_2)	d_1	m_1	$-$
2	(d_2, m_1, s_1)	$-$	m_1	s_1
	Outcome	d_1	m_1	s_1

(3)

Each plan is compatible, but the (d_1, m_1, s_1) outcome is not. The $\mathcal{D}$ and $\mathcal{S}$ components are determined by the decisive participants and the $\mathcal{M}$ component is determined by agreement (Pareto).

This example illustrates the phenomenon whereby, even in idealized settings where different groups strictly adhere to plans where everything should be compatible, even when decisions are made to reflect a sense of efficiency, it is possible to end up with incompatible conclusions. With my standard $\mathcal{D} \times \mathcal{M} \times \mathcal{S}$ example, an illustration of Thm. 2 is where it might not be possible to be manufacture a product as designed, or the design might create difficulties in sales.[7]

This example also captures the kind of interaction effect, described above, that is needed to circumvent the theorem; namely, if the two participants coordinate over the choice to be made for their respective component, a reasonable outcome emerges; e.g., it may require a "second-best" choice for some component to achieve a reasonable outcome for the system.

4 Path dependency and other consequences

It is reasonable to expect that these promised incompatible (or inefficient) outcomes occur only in highly complex settings as manifested by strong differences among the plans. This expectation is what makes the proof of Thm. 2 (Sect. 7) disturbing; the proof illustrates that incompatibilities can arise even with situations so highly consistent that the plans deviate only slightly from unanimity! (In the proof, all but one participant agree on all aspects of the plan; the deviating participant agrees on all but two of the elements.) In other words, even though two sets of inputs, or data, or conditions could appear to be essentially the same, the first set might lead to success with the divide-and-conquer approach, while the second does not. Expect even greater disarray to arise in more realistic situations that involve several decisive participants.

Thus, even when experts or specified physical laws and forces use information that carefully satisfies the compatibility conditions, it is possible to have a disordered outcome; this outcome may be manifested in terms of inefficiencies, incorrect interpretations of physical effects, or failure. Also, because such incompatibilities need *not always* occur, a particular methodology could enjoy a series of satisfactory outcomes leading to a mistaken opinion that a physical law connecting scales, an organizational principle, or a multiscale/multilevel relationship has been established. But this *need not* be true for a technique that satisfies Thm. 2. A concern is that when a guaranteed incompatible outcome occurs, blame might be mistakenly placed on faulty data or inputs rather than the true source —the nature of aggregation/multilevel rules.

As asserted, a way to avoid the Thm. 2 problem is to impose more stringent constraints on how to determine the outcome for each component; e.g., a greater level of interaction, communication, or exchange of information among the participants is required. Rather than relying on just an expert's choice, for instance, a component's outcome could be based on information coming from the plans of *all* participants. But a satisfactory methodology must be more subtle because even here problems can arise. For a simple example consider the three component situation of $\mathcal{D}, \mathcal{M}, \mathcal{S}$, where each component has two entries. The eight possible combinations are

$$(d_1,m_1,s_1),\ (d_1,m_2,s_1),\ (d_1,m_1,s_2),\ (d_1,m_2,s_2),$$
$$(d_2,m_1,s_1),\ (d_2,m_2,s_1),\ (d_2,m_1,s_2)\ (d_2,m_2,s_2).$$

The *smallest* set of admissible plans (Sect. 7), denoted by $\mathcal{A}^s$, consists of three plans with the $\{(d_2,m_1,s_1),(d_1,m_2,s_1),\ (d_1,m_1,s_2)\}$ format; the five remaining combinations are incompatible. The largest possible choice, call it $\mathcal{A}^l$, consists of six plans where (with a relabeling of the indices if necessary) the only incompatible plans are (d_1,m_1,s_1) and (d_2,m_2,s_2). Indeed, (Sect. 7), with a relabeling of subscripts if necessary, any $\mathcal{A}$ satisfying $\mathcal{A}^s \subset \mathcal{A} \subset \mathcal{A}^l$ is a set of admissible plans.

Suppose the plans proposed by design, manufacturing, and sales are as in Eq. 4: to ensure that each participant's plan influences the outcome for each component, determine the choice with a majority, or a two-thirds vote.[8] The incompatible outcome is as described in the last row:

Component	$\mathcal{D}$	$\mathcal{M}$	$\mathcal{S}$
Design	d_1	m_1	s_2
Manufacturing	d_2	m_1	s_1
Sales	d_1	m_2	s_1
Outcome	d_1	m_1	s_1

(4)

These plans come from $\mathcal{A}^s$, where $\mathcal{A}^s \subset \mathcal{A}$, so this incompatibility holds for all choices of $\mathcal{A}$. This incompatibility or inefficiency is not restricted to averaging rules; it holds for most ways to aggregate. This conflict arises because the decision for each component is made independent of the other decisions; thus these aggregation rules suffer a problem similar to that described in Thm. 2.

A troubling result comes from the "step-by-step process," which is a natural "division of labor" way to create an outcome. This is where a reasonable choice is found for as many components as possible and the choices for remaining

[7] In practice, when such an event occurs, an ajustment is made to create a compatible outcome. But, the path dependency concern raises the issue whether the adjustment is an appropriate one. Also, the adjustment involves an extra step, which increases costs, and reflects the "inefficiency" message of Thm. 2.

[8] "Majority vote" is used only to illustrate; the same effect occurs with other paired comparison rules.

components are adjusted to ensure a compatible combination. Notice how this approach describes a form of the earlier adjustment or "muddling" procedure used to complete a system analysis after natural conclusions are found for several of the parts. For instance, after finding a reasonable design choice for $\mathcal{D}$, then, if necessary, appropriately modify the manufacturing and sales plans to ensure a compatible combination; e.g., if $\mathcal{A}^s$ defines the set of compatible outcomes, selecting d_2 requires selecting m_1 and s_1. A larger set of compatible options, such as given by $\mathcal{A}^l$, offers flexibility in achieving optimality; e.g., the extra choices in $\mathcal{A}^l$ permit using any desired entry from $\mathcal{D}$ and $\mathcal{M}$, but once these choices are made, they may require an adjustment in the sales plan to achieve a compatible combination.

While "step-by-step" strategies and their variants are not unusual, expect them to generate inefficiencies, which, for engineering, may translate into lost profits or incorrect connections among scales. In particular, these strategies admit path-dependency problems with the inherent inefficiencies or incorrect conclusions. To demonstrate with a simple example, it can be shown (using ideas in Sect. 7) that with seven components and a possible relabeling of the indices an $\mathcal{A}$ includes combinations where all but one subscript is "1" or a "2;" e.g., the following three plans are in $\mathcal{A}$ where $(a_1,b_1,c_1,d_1,e_1,f_1,g_1)$ is an incompatible combination. (The choices mimic Eq. 1.)

$$\begin{array}{ccccccc} a_1 & b_1 & c_1 & d_1 & e_1 & f_1 & g_2 \\ a_2 & b_1 & c_1 & d_1 & e_1 & f_1 & g_1 \\ a_1 & b_2 & c_1 & d_1 & e_1 & f_1 & g_1 \end{array} \qquad (5)$$

Select an order over which choices are made; over each component, make an "reasonable" choice represented here by a majority vote. For any remaining components, use the "muddling approach" of making choices to ensure a compatible outcome. As with Eq. 1, by using different orders in which this "step-by-step" decision process is carried out, *seven* different outcomes emerge; they differ by which alternative has the subscript 2! For instance, if the process starts with selecting an a_j and works down through the alphabet, the selected entries are a_1,b_1,c_1,d_1,e_1,f_1 until the g_j choice is to be made. To be compatible, the g_j choice must be g_2 leading to the $(a_1,b_1,c_1,d_1,e_1,f_1,g_2)$ outcome. To have an outcome where c_2 must be selected, use the (d,e,f,g,a,b,c) order; the outcome is $(a_1,b_1,c_2,d_1,e_1,f_1,g_1)$. It is highly unlikely that all seven different outcomes are "reasonable," or that all seven capture the appropriate complexity behavior, but each could be selected. The message, then, is to expect inefficiencies from step-by-step approaches because path dependency is an accompanying, unavoidable problem. (This example illustrates Thm. 2 by assigning an appropriate decisive participant for the a_j, the b_j, and the g_j outcomes; Pareto handles all others.)

5 Parts vs. whole; Arrow's Theorem

Let me stress that Thm. 2 does *not* claim there is an absence of a relationship between the macro and micro levels; it does not assert that all multilevel approaches must suffer these failings. It states that any divide-and-conquer approach with the catalogued properties cannot be expected to find these connections.

To appreciate how to avoid the consequences of Thm. 2, recall the alternative description of Arrow's Theorem where "Arrow's Theorem asserts that when a rule uses the inputs from two or more agents (or criteria), and it is based on IIA and the Pareto conditions, then there always exist situations where the true structure of the whole need not resemble the answers developed by using information from the assembly of the parts."

There are many ways to define a "whole" of societal rankings; e.g., use the plurality vote or Borda Count. (The Borda Count tallies an n-alternative ballot by assigning $n-j$ points to the j^{th} ranked alternative, $j=1,\ldots,n$.) This interpretation of Arrow's result asserts that no matter how clever the approach that is used to determine the outcome for the parts, settings *always* exist where the answers obtained for the parts disagrees with the whole. Even more; this theorem underscores the dangers of trying to establish the structure of the whole by concentrating on the "parts."

Theorem 2 reflects the same feature; the two conditions require that the outcomes for "parts" of a system are determined in a partly independent manner. An informal interpretation of Thm. 2, then, is that problems arise by trying to determine the structure, or complexity, of a multilevel system (of whatever is being considered) with a standard divide-and-conquer multilevel approach.

Shortly after the statement of Thm. 2, I assert that ways to improve these approaches must be of the type where answers for one component are based on what happens with other components. A useful analogy, which illustrates the problem and indicates what needs to be done to solve it, comes from Rubik's Cube. The macro "parts" of this cube involve the colors on each of the six faces where the objective—the compatibility condition— is for each face to have a fixed color. The micro level includes all possible mixed combinations of the individual cubes. (An incompatible setting arises by changing the colors on some cube.) What captures the sense of Thm. 2 is to try to obtain the macro solution (i.e., solve the problem) by emphasizing individual macro "parts." That is, if an approach tries to solve the problem by first solving it for a particular face, it will be counter-productive. Instead, to solve the Rubik's Cube problem, a coordinated system approach must be discovered where the choice of each rotation takes into consideration how it affects all faces. Rather than solving the problem as a collection of parts, the approach must coordinate interactions among the different faces.

Using the lessons learned from Rubik's Cube to resolve Arrow's Theorem, the goal is to replace Arrow's conditions, which require the "whole" to agree with the answers found by considering each part separately, with requirements that require answers for the separate parts to be determined in a coordinated manner. By doing so, Arrow's dictatorial conclusion is replaced with the Borda Count (Saari [9]). Similarly, efforts to replace the negative Thm. 2 with positive conclusions must also be guided by the lessons of Rubik's Cube; such efforts will require finding the appropriate coordinated interactions when determining answers for each part of a system. (A solution for pairwise comparisons is in Saari and Sieberg [15].) While answers will be specific to the concern being examined (e.g., [14]), the general principle is the coordinated action.

6 Final comment

Rather than relying upon demanding requirements, the conditions posed in Thm. 2 represent reasonably common approaches in economics, biology, physics, engineering, manufacturing, organizational design, and so forth. Outcomes often are determined by emphasizing, at some level, "parts"—perhaps as manifested by "special physical forces." Thus, inefficiency and even impossibility must be expected. (More general results are being developed with positive assertions.) Theorem 2 encourages caution. When difficulties arise, in addition to worrying whether the data is faulty, emphasis should be directed toward re-examining the methodology.

7 Proofs

Proof of Thm. 2. Suppose there are two decisive participants where the first determines $\mathcal{C}^j$ and the second determines the $\mathcal{C}^k$ element of the outcome. Select any outcome $\mathbf{O} = (o_1, \ldots, o_n)$ and an associated plan $\mathbf{p} = (c_1, \ldots, c_n)$ where both satisfy the compensative condition of Def. 1 with respect to a specified j and k; e.g., there is a c'_k and an associated o'_k so that changing c_k to c'_k and o_k to o'_k create incompatible combinations.

Plan $\mathbf{p}$ is selected by the first participant, who determines the $\mathcal{C}^j$ component, and by all other participants except for the second decisive participant. The second participant, who selects the $\mathcal{C}^k$ component, selects plan $\mathbf{p}_k$ where c_k is replaced with c'_k to create an incompatible combination, and then replaces c_j with any compensating c'_j to create a compatible plan.

For each $i \neq j,k$, the Pareto condition determines the $\mathcal{C}^i$ component for the outcome to be the common c_i leading to o_j. For the k^{th} component, the first decisive participant determines the outcome, which is o_j. But for the k^{th} component, the second decisive participant selects c'_k creating an incompatible outcome with o'_k. The same construction applies to settings with more decisive participants. □

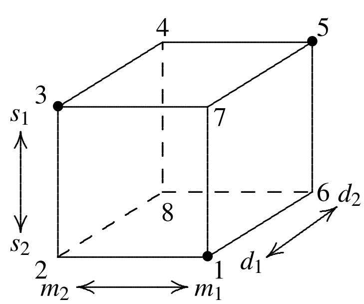

Figure 1 Three components

Proofs of the assertions made about three components. The assertions made about plans allowed with three components, say $\mathcal{D}, \mathcal{M}, \mathcal{S}$, each having two entries are verified first. The situation can be captured with the geometry of the cube in Fig. 1, where elements of $\mathcal{D}, \mathcal{M}$, and $\mathcal{S}$ are represented, respectively, by two points on the x, y, and z axes. The eight possible combinations correspond to the eight vertices. Using the names attached to the vertices, the choices are

$$7 = (d_1, m_1, s_1),\ 3 = (d_1, m_2, s_1),\ 1 = (d_1, m_1, s_2),\ 2 = (d_1, m_2, s_2),$$
$$5 = (d_2, m_1, s_1),\ 4 = (d_2, m_2, s_1),\ 6 = (d_2, m_1, s_2)\ \ 8 = (d_2, m_2, s_2).$$

As each element must be in at least one incompatible combination, the minimum number of incompatible combinations is two. To have only two incompatible combinations, one element from each of the three components must be in one combination, say (d_1, m_1, s_1) and the other elements from that set, which would be (d_2, m_2, s_2), must be in the second combination. In any selection of vertices, whenever there are precisely two incompatible combinations, they must be represented by diametrically opposite vertices.

With precisely two incompatible combinations, we must show that the six remaining combinations satisfy the compatibility conditions to be plans. First notice that they satisfy completeness and meaningfulness conditions. To verify the compensative condition, notice that a j,k condition requires finding a plan whereby a change in the j^{th} component creates an incompatible combination. Using the figure, changing the entry of a component is the same as moving from one vertex to the other on an appropriate connecting cube edge in the j^{th} direction. For instance, a $j = 2$ change is on one of the four edges parallel to the y-axis; e.g., by starting at vertex 2, the change moves to vertex 1.

For the change in a plan to result in 7, the vertex must be on an edge emanating from 7; e.g., the vertex is identified with an odd integer $\{1,3,5\}$. From 7, a change in any component returns to a plan in this set, so comprehensiveness is satisfied. Using this "L" shaped move, the set of vertices associated with 8 are the even integers $\{2,4,6\}$. As any set with the smallest number of plans must obey these L-moves relative to an incompatible combination, the compensative condition is satisfied if $\mathcal{A}$ includes either $\{1,3,5\}$ or $\{2,4,6\}$, which verifies the $\mathcal{A}^s \subset \mathcal{A} \subset \mathcal{A}^l$ assertion. A similar argument shows that the smallest $\mathcal{A}$ set with n components has n plans; e.g., with $n = 7$, one choice for the smallest $\mathcal{A}$ is the natural extension of Eq. 1.

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