PARADOXES OF VOTING

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Policy decisions such as single-sex vs. traditional marriages, funding levels for schools vs. national security, or even converting rental space to condominiums are continually being made. Rather than parsing the moral and ethical considerations, the concern addressed here is raised a level: Which group decision procedures effectively separate "right" from "wrong" outcomes? Stated more bluntly, which voting rules can be trusted?

To demonstrate the problem, suppose the preferences of a group (a "profile") selecting among Ann, Barb, and Connie are

Number	**Ranking**	**Number**	**Ranking**
4	Ann $\succ$ Barb $\succ$ Connie	5	Ann $\succ$ Connie $\succ$ Barb
2	Barb $\succ$ Ann $\succ$ Connie	3	Barb $\succ$ Connie $\succ$ Ann
7	Connie $\succ$ Barb $\succ$ Ann		

Who should win? Ann does with the plurality "vote-for-one" rule, Barb with the "vote-for-two" rule, and Connie with the Borda Count (tally ballots by assigning 2, 1 points, respectively, to the top-, second-ranked candidate): *Each candidate* wins with an "appropriate" rule! Attempting to break this impasse with head-to-head majority votes over pairs fails because Connie beats Barb, Barb beats Ann, but Ann beats Connie to create a cycle. Rules such as "Approval Voting" where a voter votes "yes or no" for each candidate, or "range voting," where a voter casts any point choice for a candidate from a specified range, generate chaotic conclusions: *any of the thirteen ways to rank three candidates is an admissible outcome;* the conclusion resembles rolling a die.

A worrisome message accompanying voting paradoxes, then, is that *election outcomes can more accurately reflect the voting rule rather than the voters' views!* This is not an isolated phenomena; for about 70% of closely contested three-candidate elections, the election ranking changes with the voting rule. This troubling situation becomes more likely with added candidates, or with approval/range voting [5].

1. Means of analysis

To address philosophical concerns, we must understand what goes wrong with voting rules and why. Academics have studied this topic starting prior to the tumultuous days of the French Revolution, where, as above, standard approaches emphasize illustrating examples. Regrettably, this methodology resembles judging a student's performance from a first quiz because, as now known (explained below), examples capturing a rule's positive aspects can be dramatically countered by other examples demonstrating negative features. As the number of examples exhibiting conflicting behaviors overwhelms anyone's ability to discover, leave alone analyze them, assertions using this highly incomplete information often are misleading.

Kenneth Arrow [1] introduced a more systematic approach of basing conclusions on "axioms." While this approach sounds promising, particularly by believing that axioms "tell us what we are getting," it has not provided the desired clarity. The reason is that as used in this field "axioms" typically are not "axioms" ([5, Chap. 3]); they are properties that only a particular rule satisfies in highly specialized circumstances, so they cannot provide insight about what happens in general.

Faced with this reality, Riker [3] described the selection of a voting rule as being subjective; thus philosophical concerns play a role in identifying subtle ethical and other considerations. Harding [2], among others, advanced this theme with the majority vote. But once mathematical advances identify voting rule structures, certain philosophical considerations become moot. Facts dismiss the need to parse the choices [4], while introducing richer directions for ethical considerations. Some of these new properties are discussed next.

2. Properties and explanations of voting paradoxes

Election rules often are measured by comparing their outcomes with those of other rules and with more/fewer candidates. Thus, an idealized objective is to characterize *everything* that could occur with any number of candidates, any combination of positional rules (i.e., tally ballots by assigning specified weights according to a candidate's position), and any profile. Surprisingly, by modifying mathematical tools from chaotic dynamics, these challenges have been answered; the "highly chaotic" conclusions [4, 5] identify all possible positional ranking paradoxes.

To provide a flavor, select and rank any number ($n \geq 3$) of candidates. Next, for each subset of candidates, select a ranking. (Thus rankings are arbitrarily selected for $2^n - (n+1)$ sets of two or more candidates; e.g., $n = 10$ defines 1,013 sets.) With the exception noted below, assign a voting rule to each subset. The conclusion: a profile exists whereby each set's sincere outcome with the assigned rule is as specified! Thus, examples exhibiting favorable consistency properties for, say, the plurality rule exist, but other examples with the Ann$\succ$Barb$\succ$Connie$\succ$Deanna plurality ranking have these voters' majority votes over pairs *reversing* this ranking and their vote-for-two outcomes over triplets reflecting the Connie$\succ$Ann$\succ$Deanna$\succ$Barb ranking!

Any ranking and most voting rules can be assigned to each set of candidates, so *any* imaginable voting paradox, no matter how perverse, exists for most voting rules! To appreciate the severity of these paradoxes, with only seven candidates, the number of lists of ranking behaviors (i.e., number of voting paradoxes) with the plurality vote exceeds 10^{50}, which surpasses the number of water droplets in the world's oceans. Only the Borda Count (BC) (assign $(n - j)$ points to the j^{th} ranked candidate) is exempt from this disturbing conclusion; e.g., BC is the *only* positional rule where its ranking reflect pairwise majority vote conclusions and how BC ranks subsets of candidates.

As these lists disclose unexpected properties of voting rules and characterize all possible ranking paradoxes, numbers such as 10^{50} underscore the futility of analyzing voting rules with properties and/or examples; something new is required. The methodology, described next with three candidates, explains all possible paradoxical ranking outcomes.

The approach [5, Chap. 4] introduces a "coordinate system" for profile space; each "direction" completely dictates what happens for indicated voting rules. "Completely" means that explanations why different rules have different rankings are "necessary and sufficient;" alternative descriptions are rewordings. This makes it immaterial whether these constructs are found to be compelling because, rather than philosophical concepts, they are mathematical necessities. Omitting a profile direction is akin to ignoring a map's North-South direction.

"Profile directions" are configurations of voter preferences that affect specified voting rules but not others. To illustrate, a configuration where one voter prefers Ann≻Barb≻Connie, while another prefers the reversed Connie≻Barb≻Ann suggests a tied outcome; it is for pairwise majority votes. For the plurality vote, however, the "Ann∼Connie≻Barb" outcome ("∼" denotes a tie) discriminates against Barb while the vote-for-two "Barb≻[Ann ∼Connie]" conclusion favors Barb; only the Borda Count ensures a tie. Surprisingly, this simple "Neutral Reversal Requirement" (NRR) [4] explains *all possible differences among positional rules*; it subsumes that large literature analyzing why these rules have different outcomes.

To illustrate the NRR power by constructing an example, if "Bob" prefers Connie≻Barb ≻Ann, different rules yield consistent election outcomes. Adding four pairs of {Ann≻Barb ≻Connie, Connie≻Barb≻Ann} and three {Ann≻Connie≻Barb, Barb≻Connie ≻Ann} to Bob's preferences retains the Borda and pairwise rankings supporting Connie, but create the conflicting plurality Ann≻Connie≻Barb and vote-for-two Barb≻Connie ≻Ann rankings. Experimenting with NRR discloses a variety of positional paradoxes and that only BC rankings never oppose pairwise outcomes.

Illustrating the next profile direction, "Neutral Condorcet requirement" (NCR), is {Ann ≻Barb≻Connie, Barb≻Connie≻ Ann, Connie≻Ann≻Barb} where, in this rotating manner, each candidate is in first-, second-, and last-position precisely once. Presumably this requires a tied outcome; it is for all positional rules. But pairwise majority votes define the cycle "Ann≻Barb, Barb≻Connie, Connie≻Ann" with 2:1 tallies. NCR directions never affect positional rankings; they cause all possible pairwise ranking anomalies [5]. To illustrate, by adding two units of {Barb≻Ann≻Connie, Ann≻Connie≻Barb, Connie≻Barb≻Ann} to Bob's preferences, the BC and all positional rankings remain fixed; the pairwise outcomes define a cycle. (To create the introductory example, add the units from these paragraphs to Bob's preferences.)

NRR and NCR directions *completely explain all possible positional/pairwise voting differences* [4, 5], so arguments supporting one rule over another must warrant their outcomes over these configurations. All possible BC and majority pairwise vote differences, for instance, are caused by the NCR direction, so arguments reduce to justifying why this NCR configuration should, or should not, cause a tie vote. Similarly, arguments for plurality, or the rule where, say, seven, two points are assigned, respectively, to a ballot's top-, second-ranked candidate, must justify why NRR configurations should not be ties and why it is acceptable for their outcomes to reverse majority rankings of pairs.

All possible three-candidate positional ranking paradoxes now are understood [5]. While profile directions for more candidates unleash wilder voting paradoxes, surprisingly they exempt BC from these inconsistencies, which explains the strong, positive BC properties. (All possible BC ranking inconsistencies are caused by NCR directions.)

3. Arrow's and Sen's theorems

Profile coordinates explain other behaviors; e.g., press articles occasionally dismiss questionable election outcomes (presumably involving voting paradoxes) with the "no voting rule is fair" phrase, which reflects the seminal Arrow impossibility theorem [1]. But profile coordinates reveal new, conflicting interpretations ([5, Chap. 2]) of Arrow's and Sen's "impossibility of the Paretian liberal" [6] theorems. .

Both influential theorems require voters with transitive preferences; both decision rules determine societal outcomes by discovering appropriate rankings for each pair. This emphasis on pairwise conclusions accurately suggests that both results are caused by NCR configurations. Thus, to analyze these theorems, seek consequences of making pairwise decisions for profiles with NCR components. The surprisingly answer ([5, Chap. 2]) is that, with both theorems, the rules dismiss the crucial requirement that voters have transitive preferences!

This observation introduces new research directions. Modifying Arrow's central property (IIA) to permit using transitivity information generates positive conclusions: Arrow's dictator is replaced with BC. Rather than Sen's assertion about constraining individual's decisions, his conclusion actually identifies when individual choices impose hardship on others. A promising new philosophical interpretation of Sen's theorem is how his result captures dysfunctional societies transitioning from one social norm to another.

4. Conclusion

This brief survey (many topics were omitted) demonstrates that while crucial, group decision processes are surprisingly complicated. Fortunately, recent results provide clarity; of particular value is how these conclusions invite re-examinations of old topics and provide new tools/themes for philosophical descriptions about group decisions.

Current mathematical conclusions strongly support using the BC for group decisions. These results reflect mathematical certainties; similar to $a^2 + b^2 = c^2$ is true for right-triangles, *in any setting even approximately covered by the assumptions of the basic theorems*, the BC is significantly superior in representing voter preferences, in avoiding paradoxes, and so forth. But several realistic settings are not approximated by these assumptions; what new concerns should be examined?

References

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[4] Saari, D. G., 2006, "Which is better: the Condorcet or Borda winner?" *Social Choice & Welfare*, **26** (1), 107-130.
[5] Saari, D. G., 2008, *Disposing Dictators, Demystifying Voting Paradoxes*, Cambridge University Press.
[6] Sen, A., 1970, "Impossibility of a Paretian Liberal," *Journal of Political Economy* **78**, 152-157.
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Suggested Reading:

(1) Emerson, P, 2007, *Towards an Inclusive Democracy*, Springer-Verlag, New York
(2) Lagerspetz, E, 1993, "Social Choice in the Real World" *Scandinavian Political Studies* **16** (1), 1-23.
(3) Lagerspetz, E, 1997, "Social Choice in the Real World II", *Scandinavian Political Studies* **20** (1), 53-67.
(4) Nurmi, H., 2002, *Voting Procedures under Uncertainty*, Springer-Verlag, New York.
(5) Saari, D. G., 2001, *Chaotic Elections! A Mathematician Looks at Voting*, American Mathematical Society, Providence R.I.
(6) Saari, D. B., 2001, *Decisions and Elections*, Cambridge University Press.