Ability Grouping in All-Pay Contests*

Jun Xiao†

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Abstract

This paper considers a situation in which participants with heterogeneous ability types are grouped into different competitions for performance ranking. A planner can allocate both the participants and a fixed amount of prize money across all-pay contests in order to maximize a weighted sum of total performance subject to individual minimal performance requirements. Both the weights and requirements are type-specific. We show that, whatever the weights and requirements are, separating – assigning participants with the same ability together – is superior to mixing – assigning participants with different abilities together. Moreover, we also characterize the associated optimal prize structures.

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†Department of Economics, University of Melbourne. E-mail: jun.xiao@unimelb.edu.au.

1 Introduction

Consider a situation in which a school wants to assign a group of students to different classrooms. Should the school group students with similar abilities together – a practice known as "tracking"– so that high ability students are separated from low ability ones, or should the school have mixed classrooms in which students of different abilities are grouped together? Tracking was very common in US schools but became less popular in the late 1980s due to the criticism of trapping students of low socioeconomic status in low-level groups. However, tracking has returned to the attention of educators recently. According to Yee (2013), "... of the fourth-grade teachers surveyed, 71 percent said they had grouped students by reading ability in 2009, up from 28 percent in 1998". Different grouping policies can be observed not only over time but also across different countries. For instance, in Germany, pupils after primary schooling are grouped into three types of secondary schools to receive training for blue-collar apprenticeships, apprenticeship training in white-collar occupations, or training for further education. In contrast, tracking was explicitly discontinued in China in 2006.[1] Not surprisingly, tracking has also been a controversial topic in the economic literature on education, and there has been a long debate on this issue from many different perspectives: students' achievement, equity, and even morality.[2]

In this paper, we examine the competitive effects of tracking and ask whether or not it enhances students' performance when their grades or rewards depend on their relative performance.[3] Two features are important to a social planner. First, zero effort or non-performance is not desirable. For instance, the planner may want every student to exert enough effort to graduate or at least attend class every-day. Moreover, the minimal performance requirements may vary across students with different abilities. These requirements impose a challenge to go beyond well-studied contest forms. For instance, if the players are identical, an all-pay auction

[1]Policies that forbade tracking in schools started in the 1990s, and a national law was passed in 2006. In contrast, tracking remains common in Chinese universities.

[2]See, for instance, Loveless (2013).

[3]We do not consider the equity issue nor the effects of tracking on the quality of instruction. If students are tracked, the classes are more homogeneous and therefore they could be easier to teach. See, for instance, Duflo et al. (2011).

maximizes the total performance (Baye et al. 1996). However, the minimal performance requirements are violated as there are equilibria in which some players exhibit non-performance.

Second, the planner may weigh students of different abilities differently. For example, a school may care more about high ability students' performance than other students. As a result, it is important to examine how different minimal performance requirements and how weights on different ability groups would affect the comparison of tracking and mixing.

Our results imply that, with a well-designed award system, tracking/separating – assigning students with the same ability together – is superior to mixing – assigning students with different abilities together. This result is true however a planner weighs different ability groups, and with whatever minimal performance requirements she has. In particular, it holds even if the planner only cares about the students of the lowest ability.

Our model also applies to the quota systems in college admissions, which let minority students compete separately for the reserved admission quota.[4] Compared to majority students, it is usually more costly for the minority students to acquire the same level of academic achievement. Therefore, the quota system separates students according to their costs, while admission without affirmative action allows all the students compete in a grand contest. Besides education, the results in this paper are also applicable to a variety of competitions. For instance, in the early nineteenth century, there were no weight classes in boxing. Then eight weight classes were introduced before the Second World War, and nine more were introduced afterwards. The history of other sports such as weightlifting and wrestling also shares similar trends. The heavier athletes have an obvious advantage in strength, so why would we want to group players with similar abilities into the same class and let them compete only within their class? This takes into consideration of the issue of fairness, and our results suggest that separating athletes according to their abilities could also increase their effort and therefore make matches more entertaining.

[4]See Bertrand et al. (2010) for more details on quota systems in India.

The key characteristics common to these scenarios are: participants with potentially different abilities/costs who are divided into different groups to compete; heterogeneous prizes awarded solely on the basis of relative performance; and sunk costs of participants' investments.

This paper builds on Siegel's (2010) model of all-pay contests by introducing heterogeneous prizes and a planner who can allocate players and prize money across contests. Specifically, suppose there is a fixed amount of prize money and a group of players with different ability types. We assume a deterministic relation between effort and performance and assume these to be equal. The players of the same type have the same constant marginal cost of performance/effort. A lower cost of performance represents a higher ability. A planner can assign the players to any number of all-pay contests and divide the money as potentially heterogeneous prizes in the contests. In each contest, the players in it choose their performance simultaneously; the player with the highest performance receives the highest prize, the player with the second highest performance receives the second prize, and so on. The planner wants to maximize the weighted sum of all players' performance with type-specific weights. Moreover, the planner can have different minimal performance requirements for different players. The minimal performance, for instance, may represent graduation requirements for students.

Our main result is that, whatever the weights are, grouping players with similar abilities together is always superior to mixing them. Moreover, we also characterize the associated optimal prize structures. Intuitively, separating leads to the most intense competition since each player has to compete against opponents who have the same ability as he does. Because of the intense competition, the players enhance their performance so much that all of them receive zero payoffs. As a result, for any set of contests with mixed players, there always exists contests of separated players with at least as much total expected performance for each ability type and a higher total expected performance for at least one ability type. In other words, having separated players dominates having mixed players in terms of the total expected performance for different types.

It is important that the planner can choose the prizes while grouping the players. In situations such as tennis or golf tournaments, choosing the prize values for different rankings is a very important part of contest design. Moreover, allowing the planner to choose prizes does not preclude the possibility that players' values of ranking are partly determined outside the tournaments. For instance, if the planner wants to have a first prize of $200K and she knows that the champion will also receive an endorsement deal worth $100K, then she can simply award the difference of $100K to the champion. A similar argument applies when a school has a budget for scholarships.

There are several challenges that we have to overcome in order to establish these results. First, with asymmetric players and heterogeneous prizes, the equilibrium of an all-pay contest may involve complicated mixed strategies. To our knowledge, equilibrium characterization for general asymmetric contests with heterogeneous prizes is still an open question. For instance, Bulow and Levin (2006), in a setting of labor market matching, allow prizes with constant differences; Siegel (2010) studies contests with identical prizes; Xiao (2013) examines quadratic (the second-order difference in prizes is a positive constant) or geometric (the ratio of successive prizes is a constant) prize sequences. In contrast, this paper allows any prize structure. The techniques developed in this paper allow us to show that contests with asymmetric players are never optimal. As a result, although we do not know the equilibrium characterization, we can still characterize the optimal way to group players.

Second, there could be multiple equilibria, and equilibrium selection may change the comparison between separating and mixing. Multiple equilibria are demonstrated in similar settings (Baye et al. 1996, Barut and Kovenock 1998). In particular, Xiao (2013) provides an example with exogenous prizes which shows that, depending on the equilibrium selection, separating may result in higher or lower total expected performance than mixing. The results in this paper apply to all equilibria, and therefore are robust to equilibrium selection.

Finally, the generality of the model also imposes extra challenges. The current paper does not restrict the number of contests, the prize structures, or the player composition in contests, which means the planner has to compare a large number of choices. Moreover, this paper accommodates a very general objective function for the planner: she can assign asymmetric type-specific weights to the players' performance, and she can impose different minimal performance requirements on different types.

Literature Baye et al. (1993) show that a politician can extract higher rent by excluding the lobbyists, who have higher valuations of winning, from the competition. This is referred to as the "exclusion principle", and it applies to many situations besides lobbying. However, our paper considers different scenarios, in which one of the planner's objectives is to ensure minimal performance levels are met for the players. Therefore, excluding a player is not optimal as it would result in zero effort/non-performance from him. Moreover, they consider a model of an all-pay auction in which the lobbyists compete for a single prize. In contrast, a single prize also leads to violation of the minimal performance requirements in our setup. According to our optimal prize structure, we need multiple potentially heterogeneous prizes to provide incentive for all the players in a contest, so each player's performance is above his particular minimal requirement.[5]

Moldovanu and Sela (2006) compare different ways to group ex ante symmetric players across contests. They find that total expected effort is maximized by a grand contest including all players, while the expected highest effort is maximized by splitting players into multiple contests and letting winners of each contest compete in a final contest. Fu and Lu (2009) find that merging multi-winner contests of symmetric players can increase total expected effort. In contrast, this paper considers players with asymmetric abilities. This assumption is crucial as we are considering whether players should be separated according to their abilities.

In the education literature, Lazear (2001) studies tracking when students are awarded according to their absolute performance. He shows that tracking results in higher total performance than mixing. Studies on peer effects also discuss grouping players across competitions (see, for instance, Board 2009 and Cooley 2009). The players' payoffs in these papers also depend on their absolute performance, while

[5] Remark 2 illustrates this piont in an example.

we consider the situation in which students are awarded according to their relative performance, or the ranking of their performance instead. Hickman's (2009) analysis on different affirmative action policies on college admissions is also related. He considers a different setup in which minority students have private costs that are stochastically higher than those of majority students, and they compete for prizes/seats of exogenous values. In a quota system, the two groups compete separately for their respective reserved seats. Compared to the case without affirmative action, the quota system has unclear effects on average performance within groups and within the overall population. In contrast, our paper considers the joint decision of grouping students and allocating prizes/resources and finds that separating leads to higher performance than mixing.

The literature on status competition studies the optimal way to divide players into different status categories when the players' payoffs depend directly on their status (see for instance Moldovanu et al. 2007 and Dubey and Geanakoplos 2010). In our paper, performance ranking does not affect players' payoffs directly. Instead, the payoffs depend on the prizes awarded according to the players' ranking.

The remainder of the paper is organized as follows. Section 2 provides a simple example illustrating how separating is superior to mixing. Section 3 introduces the model and Section 4 presents the main results on optimal grouping and optimal prize allocations. Section 5 shows that the main results are robust to small cost differences within each type, and Section 6 concludes.

2 An Example

Consider a scenario with four players and one unit of prize money. Two of the players are of H-type and the others are of L-type. The H-type players have a marginal cost of $c_H = 1$, and the L-type players have a marginal cost of $c_L = 2$.

Suppose that the players are mixed such that there is one H-type and one L-type player in each contest. Each contest has only one prize of value 1. The two players in a contest compete for the prize in an all-pay contest. More precisely, each player chooses a performance level simultaneously, and the prize goes to the

	Contest	Payoffs	Performance of a H-type player	Performance of a L-type player
Mixing	Players: H, L Prize: 1	$u_H = 1/2$ $u_L = 0$	$3/4 - u_H = 1/4$	$1/8$
Separating	Players: H, H Prize: $3/2$	$u'_H = 0$	$3/4 - u'_H = 3/4$	
	Players: L, L Prize: $1/2$	$u'_L = 0$		$1/8$

Table 1: Mixing vs. Separating

player with the higher performance. A player's expected payoff equals his expected winnings – the expected value of the prize one may receive – net of his expected cost of performance.

As we will show later, the expected winnings in equilibrium are $w_H = 3/4$ for a H-type player and $w_L = 1/4$ for an L-type player, and the expected payoffs are $u_H = 1/2$ and $u_L = 0$. Notice that one's payoff is his expected winnings net of his expected cost, so a H-type player's expected cost is $w_H - u_H$, his expected winnings minus his payoff. Since the expected cost equals the product of his marginal cost and his expected level of performance, the expected level of performance is $(w_H - u_H)/c_H = 1/4$ for a H-type player and $(w_L - u_L)/c_L = 1/8$ for an L-type player. Since $u_H = 1/2 > 0$, the H-type players receive positive rent in the equilibrium.

Now suppose the players are separated. Then, the H-type players are in one contest and the L-type players are in the other. We can verify that the payoffs are zero for all players. Suppose the prize is 0.5 for the L-type contest and 1.5 for the H-type contest. As a result, each H-type player's expected performance is $(3/4 - 0)/c_H$. Note that the expected performance is higher than $(3/4 - u_H)/c_H$ – his expected performance in a contest with mixed types – because the H-type player receives zero rent, with $u_H = 0$. Therefore, the total expected performance is $0.5/c_L = 1/4$ for L-type players and $1.5/c_H = 3/2$ for H-type players. The equilibrium outcomes with mixed and separated players are summarized in Table 1.

As illustrated in the table, the performance levels in the mixed contests are *dominated* by those in the contests with identical players in the sense that the total expected performance of L-type players is unchanged and the total expected performance of H-type players is lower in mixed contests. The remainder of the paper generalizes these insights as follows: if a planner wants to maximize the weighted average performance of different types subject to minimal performance requirements, separating is superior to mixing for any weights and minimal performance requirements the planner has.

3 Model

There is one unit of prize money and a set of players, N. The players are of T different types. Each type t has n_t players. We assume $n_t \geq 2$ for all t, so there are similar players of each type.[6] There is a deterministic relation between effort and performance, and we assume these to be equal. Players of t-type have the same marginal cost of performance, c_t. Without loss of generality, we assume $0 < c_1 < c_2 < ... < c_T$.

A planner's decision has two parts: assigning the players into any number of contests, and dividing the prize money as prizes for each of the contests. More precisely, the planner's choices can be represented by a partition of the players $\mathcal{P}$, and a prize structure $\mathcal{V}$. The partition $\mathcal{P}$ is a family of non-empty subsets of N such that N is a disjoint union of the subsets. Suppose partition $\mathcal{P}$ consists of m sets, then we denote $\mathcal{P} = \{P_1, ..., P_m\}$. The prize structure $\mathcal{V}$ is a family of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_m\}$, where vector $\mathbf{v}_k \in [0, 1]^{\#P_k}$ and $\#P_k$ is the number of players in set P_k. A zero entry of $\mathbf{v}_k$ means one of the prizes is zero. Therefore, the partition $\mathcal{P}$ and prize structure $\mathcal{V}$ characterize m individual contests. In particular, the subset $P_k \subset N$ is the set of players assigned to contest k, and $\mathbf{v}_k$ represents the prizes in contest k. Since there is no restriction on the number of contests, the planner may assign all the players into one contest, that is, $\mathcal{P} = \{N\}$.

Let us describe the competition in each contest. In a contest characterized by P_k and $\mathbf{v}_k$, all the players in P_k choose their performance/effort levels in $[0, +\infty)$ simultaneously. The player with the highest performance receives the highest prize in $\mathbf{v}_k$; the player with the second highest performance receives the second highest

[6] This assumption is relaxed in Section 5.

prize in $\mathbf{v}_k$; and so on. In the case of a tie, the prizes are allocated randomly such that no tying player loses with certainty.[7] If a player wins a prize, his payoff is his prize net of his cost of performance. If a player wins no prize, his payoff is zero minus his cost of performance. All players are risk neutral. This paper considers Nash equilibria. A profile of strategies constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players.

The planner's objective can be characterized by two sets of parameters: weights (α_1, ..., α_T) and minimal performance (r_1, ..., r_T). First, the planner wants to maximizes a weighted sum of expected performance, and she attaches weight α_i to a t-type player i's expected performance. We assume that $\alpha_i \geq 0$ and $\Sigma_{i=1}^{n} \alpha_i = 1$. The weight α_t represents the relative importance of t-type players to the planner, and she only cares about the performance of t-type if $\alpha_t = 1$. Second, the planner wants the players of all types to exhibit some performance in competition. That is, a t-type player's expected performance should be at least $r_t > 0$. The requirements can represent the minimal performance to graduate, or the requirement to attend class every day. Note that values in $r_1, ..., r_T$ need not be the same.

Let $E[s_{i_t}]$ denote the equilibrium expected performance of player i_t of t-type. The planner chooses partition $\mathcal{P}$ and prize structure $\mathcal{V}$ to maximize the weighted sum of expected performance subject to the minimal performance requirements. Therefore, her problem is

$$\begin{aligned} \max_{\mathcal{P},\mathcal{V}} \quad & \sum_{t=1}^{T} \left(\alpha_t \sum_{i=1}^{n_t} E[s_{i_t}] \right) \\ \text{s.t.} \quad & E[s_{i_t}] \geq r_t \text{ for } i_t = 1, ..., n_t \text{ and } t = 1, ..., T. \end{aligned} \tag{1}$$

If $\Sigma_t r_t / c_t > 1$, the minimal performance requirements are never satisfied because of limited prize money. As a result, we assume that the minimal performance requirements are not too high. That is, $\Sigma_t r_t / c_t \leq 1$.

[7] In many tournaments (for example, in golf), ties are resolved by sharing the prizes. As an example, if two players tie with the second-highest score, then each receives the average of the second and third prize. Our formulation allows this kind of sharing.

4 Optimal Grouping

If a contest only has one player, his equilibrium performance level must be zero, which violates his minimal performance requirement. Therefore, we only need to consider the partitions in which each contest contains at least two players. The planner is said to *separate* the players if each contest contains players of a same type. Otherwise, we say the planner *mixes* the players. Note that the planner needs at least T contests to separate the players, but she can have more than T contests by splitting a larger contest into smaller ones. The main result of this paper is as follows.

Theorem 1 *For any given weights and minimal performance levels, the total weighted expected performance is maximized only if the players are separated.*

In the remainder of this section, we prove Theorem 1 through a sequence of lemmas and specify the associated optimal prize structure in Proposition 1. The first lemma ensures that each contest has an equilibrium.

Lemma 1 *In each contest, there exists no Nash equilibrium in pure strategies, but there exists a Nash equilibrium in mixed strategies.*

The proof of this lemma is in the Appendix. In contrast to Siegel's (2010) existence proof for contests with homogeneous prizes, this lemma ensures equilibrium existence in asymmetric contests with heterogeneous prizes.

Consider a contest with a player set P_m and a prize vector $\mathbf{v}_m$. If a player i chooses performance level s, his expected winnings $W(\mathbf{G}_{-i}(s), \mathbf{v}_m)$ depend on the strategies of others $\mathbf{G}_{-i}(s) \equiv (G_j(s))_{j \in P_m \setminus \{i\}}$ and the prizes $\mathbf{v}_m$.

Lemma 2 *Given any equilibrium in a contest with different prizes, let $\bar{s}_j$ be player j's highest performance in the support of his equilibrium strategy. Then,*

$$W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_m) \geq W(\mathbf{G}^*_{-j}(\bar{s}_j), \mathbf{v}_m), \tag{2}$$

where $W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_m)$ *is player* i*'s expected winnings if he chooses* $\bar{s}_j$ *while others choose the equilibrium strategies* $\mathbf{G}^*_{-i}(s)$*, and* $W(\mathbf{G}^*_{-j}(\bar{s}_j), \mathbf{v}_m)$ *is player* j*'s expected winnings if he chooses* $\bar{s}_j$ *while others choose the equilibrium strategies* $\mathbf{G}^*_{-j}(s)$*.*

Proof. There are two possibilities. First, suppose $\bar{s}_j = 0$. It means player j chooses performance level 0 with probability 1, then Lemma 5 implies that player j loses with certainty. Hence $W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_m)$ cannot be lower than j's expected winnings in the equilibrium. That is, inequality (2) holds.

Second, suppose $\bar{s}_j > 0$. Lemma 5 implies that no player chooses $\bar{s}_j$ with positive probability in the equilibrium. Therefore, if player i chooses performance level $\bar{s}_j$, his performance is higher than j's with certainty, so player i never receives the lowest prize. As a result, player i's expected winnings $W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_m)$ is the same as if player j and the lowest prize were excluded. More precisely, if we exclude player j and the lowest prize, the new contest has player set $P_m \backslash \{j\}$ and prize vector $\mathbf{v}'_m$, which is $\mathbf{v}_m$ with the lowest prize removed. Let $\hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m)$ be i's expected winnings in the new contest if the other players are choosing strategies $\mathbf{G}^*_{-i-j}(s) \equiv (G^*_k(s))_{k \in P_m \backslash \{i,j\}}$. Then we have

$$W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_m) = \hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m). \tag{3}$$

Recall that player i does not choose $\bar{s}_j$ with positive probability. If player j chooses performance $\bar{s}_j$, player i's performance level s_i is strictly below $\bar{s}_j$ with probability $G^*_i(\bar{s}_j)$. Similar to the above argument, player j's expected winnings by choosing $\bar{s}_j$ conditional on $s_i < \bar{s}_j$ is $\hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m)$. With probability $1 - G^*_i(\bar{s}_j)$, player i's performance level s_i is strictly above $\bar{s}_j$. Player j's expected winnings by choosing $\bar{s}_j$ conditional on $s_i > \bar{s}_j$ is $\hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}''_m)$, where $\mathbf{v}''_m$ is $\mathbf{v}_m$ with the highest prize removed. Therefore, we can rewrite j's expected winnings at performance $\bar{s}_j$ as

$$\begin{aligned}
& W(\mathbf{G}^*_{-j}(\bar{s}_j), \mathbf{v}_m) \\
= \ & G^*_i(\bar{s}_j)\hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m) + (1 - G^*_i(\bar{s}_j))\hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}''_m) \\
= \ & \hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m) \\
& -(\hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m) - \hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}''_m))(1 - G^*_i(\bar{s}_j)). \qquad (4)
\end{aligned}$$

Without loss of generality, suppose the entries in $\mathbf{v}'_m$ and $\mathbf{v}''_m$ are ranked from the highest to the lowest. Since the prizes in $\mathbf{v}_m$ are not identical, all entries of $\mathbf{v}'_m - \mathbf{v}''_m$ are non-negative, with at least one entry is positive. As a result, the expected winnings given the same strategies is also higher, that is, $\hat{W}(\mathbf{G}^*_{-i-j}(s), \mathbf{v}'_m) > \hat{W}(\mathbf{G}^*_{-i-j}(s), \mathbf{v}''_m)$. Therefore, (4) implies

$$W(\mathbf{G}^*_{-j}(\bar{s}_j), \mathbf{v}_m) \geq \hat{W}(\mathbf{G}^*_{-i-j}(\bar{s}_j), \mathbf{v}'_m) = W(\mathbf{G}^*_{-i}(\bar{s}_j), \mathbf{v}_m),$$

where the equality comes from (3). Hence, if $\bar{s}_j > 0$, we also have (2). ■

One of the challenges is that there may be multiple equilibria in a contest. Our method relies only on the properties that are true for any equilibrium, thus we overcome this challenge. The lemmas below present two such properties of any equilibrium.

Lemma 3 *If all the players in a contest are identical, each player's expected payoff in any equilibrium must equal to the value of the lowest prize.*

Proof. Given any equilibrium, let $\underline{s}$ denote the lowest performance level in the supports of the mixed strategies. Then, at least one player's mixed strategy has $\underline{s}$ as the lower bound of its support. If $\underline{s} > 0$, this player wins the lowest prize with performance $\underline{s}$. On the other hand, he could also win the same prize with performance 0, which incurs a lower cost. This is a contradiction. As a result, we must have $\underline{s} = 0$, and the payoff of this player equals the lowest prize. We will show below that no player can have a different payoff.

If the prizes are identical, everyone receives the same prize hence the lemma is true. Suppose the prizes are not identical, and suppose player i's payoff equals the lowest prize and player j's payoff in an equilibrium is higher than the lowest prize. Let $\bar{s}_j$ be the highest performance level in the support of player j's mixed strategy and $\bar{s}_i$ be the counterpart for i. Lemma 2 implies that player i's expected winnings with performance level $\bar{s}_j$ is no lower than player j's. In addition, notice that both players have the same marginal cost, so player i's payoff at $\bar{s}_j$ is no lower than that of j at $\bar{s}_j$. This contradicts the assumption that player j's payoff is higher than i's. ■

Lemma 4 *If the players in a contest are not identical and the prizes are different, at least one player's payoff is higher than the lowest prize in any equilibrium.*

Proof. Suppose player i and j have different costs, with $c_i > c_j$. Since the payoffs cannot be lower than the lowest prize, it is sufficient to show that players i and j have different payoffs. Assume otherwise that $u_j = u_i$, where u_i and u_j are i and j's expected payoffs in an equilibrium. According to Lemma 2, player j's expected winnings at $\bar{s}_i$ is no lower than i's at $\bar{s}_i$. Therefore, if player j deviates to $\bar{s}_i$, his expected winnings is not lower than i's, but his cost is lower than i's. Hence, the deviation results in j's payoff higher than u_i. This is a contradiction because i and j have the same payoff by assumption. ■

Now we can proceed to prove Theorem 1. Denote $S_t \equiv \sum_{i=1}^{n_t} E[s_{i_t}]$ as the total expected performance of all t-type players. Roughly speaking, we show below that any performance outcome $(S_1, ..., S_T)$ with mixed players is *dominated* by an outcome $(S'_1, ..., S'_T)$ with separated players in the sense that $S_i \leq S'_i$ for all i and $S_i < S'_i$ for some i. This shows that separating is superior to mixing for any type-specific weights.

Proof of Theorem 1. Suppose the players are mixed. Consider an equilibrium in which the minimal performance requirements are satisfied. That is, $E[s_{i_t}] \geq r_t$ for any type t. Given the equilibrium, let $(S_1, ..., S_T)$ be the performance outcome, U_t be t-type players' total expected payoff, and W_t be their total expected winnings.

For each player, his payoff equals his expected winnings net of the cost of his expected performance, so we have $U_t = W_t - c_t S_t$, which gives us the expression of his expected performance

$$S_t = (W_t - U_t)/c_t. \tag{5}$$

Note that U_t is the total payoff when the players are mixed. Lemma 4 implies that $U_t > 0$ for some t, so we also have

$$\Sigma_t(W_t - U_t) < \Sigma_t W_t = 1. \tag{6}$$

Suppose that the planner separates the players into T contests, so each contest contains all the players of a particular type. In addition, suppose she assigns prizes of total value $W_t - U_t$ to the contests with t-type players and all the prizes except the lowest are positive. Then, the total winnings of t-type players equals the total value of their prizes, $W_t - U_t$. If the lowest prize is zero in every contest, Lemma 3 implies that all players have zero payoffs. Similar to (5), the total expected performance of t-type players is $((W_t - U_t) - 0)/c_t = S_t$, and the performance outcome is the same as $(S_1, S_2, ..., S_T)$. According to (6), some prize money, $1 - \Sigma_t(W_t - U_t)$, is not assigned to any contest. If the planner adds the extra money to the first prize in a contest of t-type players, the total performance of t-type would increase. Hence, the resulting performance outcome dominates $(S_1, S_2, ..., S_T)$.

To complete the proof, we still need to ensure that the minimal performance requirements are satisfied. Recall that in the equilibrium with mixed players, we have $E[s_{i_t}] \geq r_t$ for any type t. Denote i'_t as the player with lowest expected performance among the t-type players in the equilibrium with mixed players, so his expected performance is

$$E[s_{i'_t}] \geq r_t. \tag{7}$$

Therefore,

$$S_t \geq n_t E[s_{i'_t}]. \tag{8}$$

According to Barut and Kovenock (1998), if all the players in a contest are identical and all the prizes except the lowest are positive, the contest has a unique

Nash equilibrium in which the players have symmetric strategies. Since all t-type players are grouped in the same contest, they have the same expected performance. Thus each of them has expected performance $S'_t/n_t \geq S_t/n_t \geq E[s_{i'_t}] \geq r_t$, where the first inequality comes from the dominance shown above, the second comes from (8), and the last comes from (7). Hence, individual minimal performance requirements are also satisfied. ■

Remark 1 *According to the theorem, it is never optimal to assign only one player to a contest or to have a contest containing all the players. In addition, even if the planner only wants to maximize the performance of one type, it is still optimal to separate the other players. This is because, if the players are separated, less prize money is needed to ensure the minimal performance requirements are met by the other players.*

It is also worth mentioning that there is more than one way to separate the players. For instance, suppose there are two H-type players and four L-type players. The planner can separate the players in two ways. She can have two contests with all the H-type players in one and all the L-type players in the other. Alternatively, she can have one contest with all the H-type players and two other identical contests, each with two L-type players. Proposition 1 below implies that the optimal performance outcome remains the same across the different ways of separating.

Now let us consider the optimal prize structure. As a result of Theorem 1, we only need to find the optimal prizes for separated players. According to Lemma 3, if the lowest prize in a contest becomes smaller, the total expected payoff in the contest decreases, therefore the total expected performance increases because of (5). Hence, the lowest prize should be zero in every contest, then all players should have zero payoffs. Therefore, equation (5) implies that the total performance of t-type players is V_t/c_t, where V_t denotes the total value of prizes assigned to the contests containing t-type players. Note that, if the players are separated, the distribution of prize money within a contest has no effect on the total performance in the contest as long as all the prizes except the lowest remain positive.[8] As a result, the planner's problem (1) becomes a linear programming problem

$$\begin{aligned} \max_{V_1,..,V_T \geq 0} \quad & \Sigma_t \alpha_t V_t / c_t \\ \text{s.t.} \quad & \Sigma_t V_t = 1, \\ & V_t/(c_t n_t) \geq r_t \text{ for all } t. \end{aligned}$$

If $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$, it is optimal to minimize V_t, so $V_t = c_t n_t r_t$ is just enough to maintain the minimal performance requirement. In addition, we also need to ensure the minimal effort requirements are met. Recall that each player in a contest of identical players exhibits the same expected performance. Therefore, in order to ensure the minimal performance requirements are met, the per capita prize should be at least $c_t r_t$ in each contest containing t-type players. Based on the analysis above, the proposition below characterizes the optimal prize structures that solve the planner's problem.

Proposition 1 *A prize structure is optimal for separated players if and only if i) all the prizes except the lowest are positive in each contest, ii) the total value of prizes in all the contests of t-type players is $V_t = c_t n_t r_t$ if $\alpha_t/c_t < \alpha_{t'}/c_{t'}$ for some $t' \neq t$, and iii) the total value of prizes in a contest containing k_t t-type players is at least $c_t k_t r_t$.*

Remark 2 *A single prize, as in an all-pay auction, is not optimal if a contest contains more than two players. For instance, if there are three H-type players and two L-type players, the only way to separate them is grouping all the H-type players in one contest and both L-type players in another. Then, if we assign a single prize in the contest with three H-type players, there exists an equilibrium in which one player chooses zero performance level. Therefore, the minimal performance requirement is violated.*

[8] In different setups where the participants' costs are private information, allocation of prizes would affect the equilibrium performance. See, for example, Moldovanu et al. (2007) and Liu et al. (2013).

There could be more than one optimal prize structure, but they all result in the same performance outcome. Moreover, because Lemma 3 applies to any equilibrium, the optimal prize structures and the induced performance outcome remain the same whether there are multiple equilibria or not.

So far we have demonstrated that for separating to be optimal, it should be accompanied with associated optimal prize structures. The share of the budget that is used to motivate the players of t-type is weakly increasing in the weight α_t or the minimal performance level r_t of this type. If the planner cannot choose the prizes freely, mixing could actually be better than separating. For example, if the school cannot ensure enough scholarships or resources are allocated to the lower-ability groups, the students in those groups could have academic achievements below the minimal requirements. Then, it could be beneficial to mix the students with different abilities.

5 Robustness

Section 3 and 4 consider the case in which the players of the same type have identical marginal costs. This section relaxes the assumption and shows that our main result, Theorem 1, is robust to small idiosyncratic shocks in the costs. More precisely, suppose player i_t of t-type has a marginal cost $c_t + \varepsilon_{it} > 0$, where ε_{it} represents the idiosyncratic shock. Note that the game is of complete information, so the shocks are commonly known by the players. If the shocks are small, t-type players' marginal costs are close to c_t.

Proposition 2 *Suppose the cost differences across players of the same type are small, that is,* $\max_{i,t} |\varepsilon_{it}|$ *is close to zero. For any weights and minimal requirements, the total weighted expected performance is maximized only if the players are separated.*

The proof of the proposition is in the Appendix. Roughly speaking, Lemma 3 and 4 still apply if the cost differences converge to zero, so Proposition 2 can be proved in a similar manner to Theorem 1.

The proposition implies that players with similar abilities should be grouped together. It is worth mentioning that grouping players with similar abilities together does not necessarily mean grouping players with higher abilities together. On one hand, they could represent the same allocation of players. For instance, suppose that there are four players with distinct marginal costs, where $0 < c_1 < c_2 < c_3 < c_4$. Fixing c_1 and c_4, if $c_3 - c_2$ converges to $c_4 - c_1$, player 2's marginal cost converges to 1's, and player 4's converges to 3's. Therefore, the proposition above implies that the optimal allocation groups similar players together. That is, placing players 1 and 2 in a contest and players 3 and 4 in another, which in this particular case also means the players with higher abilities are grouped together.

On the other hand, grouping players with higher abilities together does not necessarily mean grouping similar players together. Then, grouping players with higher abilities may not be optimal. Example 1 illustrates such a situation.

Example 1 *Consider four players with marginal costs of* $c_1 = 1$, $c_2 = 10$, $c_3 = 11$, *and* $c_4 = 20$. *The planner maximizes the total expected performance subject to the same minimal performance requirement of* $r = 10^{-4}$ *for all the players.*

According to calculations in the Appendix, the maximum total expected performance given allocation $(\{1, 2\}, \{3, 4\})$ is 0.05, while the maximum total expected performance level given allocation $(\{1, 4\}, \{2, 3\})$ is 0.08. The allocation $(\{1, 2\}, \{3, 4\})$ groups players with higher abilities together, and it seems to be a natural candidate for optimal allocation of players, but it is not optimal. This is because, though player 1 and 2 have lower marginal costs, the difference between their costs is the smallest.

6 Conclusion

This paper studies how to group players of different abilities across all-pay contests when the prize structure is endogenous. We demonstrate that separating the players according to their abilities is superior to mixing them, and we also characterize the associated optimal prize structures. It would be an interesting extension to consider the optimal grouping problem if players are all different. As in Example 1, the optimal grouping would depend on the distribution of costs. Moreover, since each contest inevitably has asymmetric players, it is very important to characterize the equilibria in asymmetric contests with heterogeneous prizes, which to our knowledge is still an open question. Similarly, extending the model to accommodate constraints on the maximum number of contests or on the number of participants in each contest would lead to the same challenge in equilibrium characterization.

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Appendix

The lemma below characterizes an important property of atoms in one's equilibrium strategy, and we use this lemma to prove Lemma 1.

Lemma 5 *Suppose a player has an atom at performance level s in an equilibrium, that is, he chooses s with a strictly positive probability. Then he loses with certainty by choosing this performance level.*

Proof. We first claim that, if two or more players have an atom at performance level s in an equilibrium, all the players who have an atom at s lose with certainty.[9] Let us prove it by contradiction. Suppose that two players, i and j, have an atom at performance level s in an equilibrium, and suppose that player i wins a prize with positive probability by choosing s. Since the tie is broken in such a way that everyone involved wins with positive probability, player j also wins a prize with positive probability by choosing s. In contrast, if player j increases his performance slightly above s, his cost is almost the same but his expected winnings would have a discontinuous increase. This is because he no longer needs to share any prize with player i. This is a deviation for player j, which is a contradiction.

We prove Lemma 5 in two steps. First, suppose two players have an atom at performance level s in the equilibrium, then the above claim implies that both of them must lose with certainty by choosing s. Second, suppose only player i has an atom at s, and suppose he wins a prize with positive probability. On the one hand, if all other players have no best response in $(s-\varepsilon, s)$ for some $\varepsilon > 0$, player i would benefit from lowering the atom to $s-\varepsilon$. This is a contradiction. On the

[9]This claim is referred to as the Tie Lemma by Siegel (2009).

other hand, suppose another player j has a sequence of best responses converging to s from below. Compared to such a best response close to s, performance slightly above s imposes an almost identical cost on player j, but the resulting expected winnings would have a discontinuous increase because of player i's atom at s. This is also a contradiction. In sum, player i loses with certainty by choosing performance level s, which completes the proof. ■

Proof of Lemma 1. Lemma 5 implies that every player loses in a pure strategy equilibrium, which cannot be true. Hence, there exists no equilibrium in pure strategies. Moreover, at most one player has an atom in an equilibrium, otherwise Lemma 5 is violated because one of the players would win with positive probability by choosing the atom performance level.

Now we show that there is an equilibrium in mixed strategies. Consider a contest with players $1, 2, ..., n$, and let v_1 be the highest prize and c_1 be the lowest cost in the contest. Then, no player chooses the performance above v_1/c_1 because it costs more than the highest prize. Let us consider a restricted action space $\Pi_{i=1}^{n}[0, v_1/c_1] \backslash \{(s_1, ..., s_n) | s_i = s_j \text{ for some } i, j\}$, in which the performance levels are distinct and between 0 and v_1/c_1. Players' payoffs are bounded and continuous in the restricted space, which is dense in $\Pi_{i=1}^{n}[0, v_1/c_1]$. According to Simon and Zame (1990, p. 864), there exists some tie-breaking rule, which may be performance dependent, such that the contest with action space $\Pi_{i=1}^{n}[0, v_1/c_1]$ has an equilibrium in mixed strategies. To complete the proof, it suffices to verify that the equilibrium in the restricted contest remains an equilibrium in the original contest. We do this in two steps.

First, a best response in the equilibrium above gives player i the same payoff in the restricted and original contests if all others follow the strategies in the equilibrium. Recall that there is no atom at a positive performance level in the original contest, and it is also true in the restricted contest. As a result, if a best response of player i in the equilibrium is positive, no other players have an atom at the best response, which means that the probability of a tie at the best response is zero. Therefore, the best response gives player i the same payoff in both the restricted and original contests. If a best response of player i in the equilibrium is zero, the player loses with certainty and gets zero payoff in both contests.

Second, no performance level gives player i a payoff in the original contest higher than his payoff in the restricted contest. If the others do not have an atom at a particular performance level, it would give player i the same payoff in both contests. As a result, we only need to check the performance levels, at which player i does not have an atom but others do. This is the performance level of 0. If another player has an atom at 0, by choosing a performance level slightly above 0, player i could get an discontinuous increase in his payoff compared to that at 0. As a result, a performance level at 0 does not give player i a higher payoff in the original contest.

■

We now proceed to prove Proposition 2. First, we show that Lemma 3 and 4 still hold if the cost differences within each type are small enough. After that, similar to proving Theorem 1, we use these two lemmas to prove Proposition 2.

Proof of Proposition 2. Let us first show the counterpart of Lemma 3: the players' payoffs in a contest converge to the value of the lowest prize in any equilibrium if $\max_{i,t} |\varepsilon_{it}|$ goes to zero. Similar to Lemma 3, the player with the highest marginal cost, say player n, has a payoff that equals to the lowest prize. According to Lemma 2, player n can ensure himself expected winnings no less than that of player i's at performance level $\bar{s}_i$, player i's highest performance in the support of his strategy. As a result, if players n and i's costs converge towards each other, player n's payoff cannot be lower than i's in the limit. Since no player's equilibrium can be lower than the lowest prize, the payoffs of player i and n must be the same and equal to the lowest prize in the limit.

Let us now show the counterpart of Lemma 4: if the players in a contest have different cost types and the prizes are not identical, at least one player's payoff is higher than the lowest prize in any equilibrium. Suppose players i and j have different cost types, then their costs in the limit are also different: $c_i > c_j$, also suppose that they have the same equilibrium payoff in the limit, that is, $u_i = u_j$. According to Lemma 2, player j can ensure himself expected winnings no lower than that of player i's at performance level $\bar{s}_i$. Therefore, player j can ensure himself a payoff higher than i's by choosing $\bar{s}_i$ because j has a lower cost in the limit. This is a contradiction.

Given the counterparts of Lemma 3 and 4, we can prove Proposition 2 in the exact same way as we prove Theorem 1. ■

Calculations for Example 1. Suppose the allocation is $(\{1,2\},\{3,4\})$. Then, player 1 and 2 compete in a contest for a prize of $v \in (0,1)$, and players 3 and 4 compete in the other contest for a prize of $1-v$. The equilibrium payoffs are $u_1 = v\left(1 - c_1/c_2\right)$ for player 1 and $u_2 = 0$ for player 2. The equilibrium strategies are $G_1(s) = (u_1 + c_1 s)/v$ and $G_2(s) = c_2 s/v$. Hence, the expected performance of the two players are

$$\begin{aligned} E[s_1] &= \int_0^{v/c_2} s dG_1(s) = v/(2c_2), \\ E[s_2] &= \int_0^{v/c_2} s dG_2(s) = vc_1/(2c_2^2). \end{aligned}$$

Similarly, the expected performance for players 3 and 4 are $E[s_3] = (1-v)/(2c_4)$ and $E[s_4] = (1-v)c_3/(2c_4^2)$.

Given the allocation $(\{1,2\},\{3,4\})$, the planner chooses a prize structure to maximize the total expected performance subject to the minimal performance requirement. That is,

$$\begin{aligned} &\max_{v\in(0,1)} \sum_{i=1}^{4} E[s_i] \\ \text{s.t.} \quad & E[s_i] > r \text{ for } i = 1,2,3,4 \end{aligned}$$

Substituting the parameter values into the problem above, we can rewrite the problem as

$$\begin{aligned} &\max_{v} \frac{11}{200}v + \frac{31}{800}(1-v) \\ \text{s.t.} \quad & 0.02 \leq v \leq 1 - 0.08/11 \end{aligned}$$

Therefore, the optimal prizes are $1 - 0.08/11$ for the contest between players 1 and 2, and $0.08/11$ for the contest between players 3 and 4. The resulting total expected performance is 0.05.

Similarly, suppose that the allocation is $(\{1,4\},\{2,3\})$, then the planner's problem can be rewritten as

$$\begin{aligned} &\max_{v} \frac{21}{242}v + \frac{21}{800}(1-v) \\ \text{s.t.} \quad &0.0121/5 \leq v \leq 0.92 \end{aligned}$$

Therefore, the optimal prizes are 0.92 for the contest between players 2 and 3, and 0.08 for the contest between players 1 and 4. The resulting total expected performance is 0.08, which is higher than the total expected performance for allocation $(\{1,2\},\{3,4\})$.