Financial Constraint and Firm Dynamic

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Abstract

This paper develops a continuous-time principal-agent model and solve a dynamic debt contract to endogenize financing constraints and studies its implications for firm dynamics as in the discrete-time model of Clementi and Hopen-hayn(2006). We develop a method to solve for the optimal contract, given the incentive constraints, in a continuous-time setting, and study the properties of the optimal dynamic debt contract, and relate them to the firm growth rate and its volatility, and survival probability. In agreement with the empirical evidence, we shows explicitly that the firm size is concave and increasing with the equity value, the growth rate and its volatility and the survival probability decrease with the firm size and age. Finally, the model is able to generate the evolution of skewness in firm size distribution documented by Cabral and Mata(2003).

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1 Introduction

Agency problems limit the extent of borrowing and consequently may be important determinants of firm dynamics. In this paper, we consider a dynamic contracting environment in which a risk-neutral entrepreneur with limited endowment of wealth has the opportunity to invest in a project. At the beginning of the project, the entrepreneur has a project that requires a fixed setup investment. Once in operation, the project generates cumulative cash flows that follow a Brownian motion with positive drift, which follows a standard production function. While the probability distribution of the cash flows is publicly known, the entrepreneur may distort these cash flows by taking a hidden action that leads to a private benefit for herself and a loss to the lender.Therefore, from the perspective of lender that fund the project's working capital, there is the concern that a low cash flow realization may be a result of agency problems (for example, the entrepreneur diverts realized cash flows as her private benefits), rather than the project's fundamentals. To provide the entrepreneur with appropriate incentives, the bank control the entrepreneur's cash flow right, and may also withdraw the firm's working capital and terminate the project, providing a fixed liquidation value.

Using techniques introduced by annikov(2008) and DeMarzo and Sannikov(2006), we develop a martingale approach to formulate the agent's incentive compatibility constraint. We then characterize the optimal contract through an ordinary differential equation. This characterization, unlike that using the discrete-time Bellman equation, allows for an analytic derivation of the impact of the model parameters on the optimal contract, firm dynamics, and well as investment sensitivity with respect to cash flows. Such kind of continuous-time setting offers several advantages. First, it provides a much cleaner characterization of the optimal contract through an ordinary differential equation. Second, it yields a explicit determination of firm dynamics, allowing us to compute comparative statics of firm dynamics.

In the absence of asymmetric information, there exists an efficient working capital investment, which is advanced every period, such that the firm neither grows nor exits. In contrast, with asymmetric information we characterize the optimal contract contingent upon the equity value (i.e., the expected discounted value of the cash flows accruing to the entrepreneur). When the entrepreneur suffers from bad lucks that the project gets continually negative shocks such that the equity value goes beyond a low boundary, the bank ceases to invest the project and the firm is liquidated. When he is lucky enough that the project gets continually positive shocks such that the equity value reaches a upper threshold, financial constraints cease to bind and the firm attains its efficient size, and the entrepreneur takes a large proportion of the firm's cash flow rights and gains a positive consumption. When the equity value goes between, the bank takes the full proportion of the firm's cash flow rights and under-invest the project, and the entrepreneur delays her consumption. The transfer policy in the optimal dynamic contract is quite similar with that of the standard short-term debt. The standard short-term debt contract specifies a contingent transfer policy that the entrepreneur gains the residual claim right when the project's cash flow is not less that the pre-specified repayment, and otherwise the lender gains the whole cash flow rights.

The optimal contract determines the nontrivial diffusion processes for firm size (the working capital investment policy), equity (the entrepreneur's share of total firm value), and debt (the lender's share). Production shocks affect the firm's equity and debt values and financial structure, and thus have persistent effects on the firm size and age, growth rate and volatility, survival probability, and investment sensitivity with respect to financial constraint. The optimal dynamic contract consists of the investment level, transfer policy, and liquidation. The transfer policy rewards and punishes to discipline the entrepreneur's conduct. This explains why firm size (the investment policy) and equity value decreases with the production volatility.

Our paper contributes to an emerging literature on financial constraint and firm growth. The conventional wisdom known as Gibrat's law asserts that firm size and growth are independent and that the firm size distribution (FSD) is stable over time and approximately lognormal. This view has been challenged by a series of recent papers (see Sutton(2001) and Lotti, Vivarelli and Santarelli(2004) for recent surveys of the literature). The empirical regularities of firm dynamics show that

- Size dependence: Conditional on age, the dynamics of firms (growth, volatility of growth, job creation, job destruction and exit) are negatively related to the size of firms
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More recently Cabral and Mata(2003) document two stylized facts about the FSD: the distribution of young firms is very skewed to the right (most of the mass is on small firms); and the skewness tends to diminish monotonically with firm age (the distribution of older firms is more symmetric than that of young firms). Next, the paper presents a simple theoretical model in which financial constraints determine the observed FSD evolution, and provides supporting empirical evidence. Angelini and Generale(2005), based on a sample containing survey-based measures of financial constraints for Italian firms, confirms the negative link between financial constraints and firm size.

These views have recently found some theoretical support. Bolton and Scharfstein(1990) consider a two-period model with asymmetric information similar to ours, but without a choice of scale. Gertler(1992) is a first move in this direction and studies the optimal contract between a lender and a borrower in a three-period production economy with asymmetric information. Clementi and Hopenhayn(2006) extends Gertler's to infinite-period one and builds a clearly dynamic model, and matches most of the qualitative properties of firm dynamics that have been recently documented. Due to the calculation complexity of discrete model, they consider only two contingent state of outcome and the lender takes random policy to terminate the project with a probability. In spite of its simplification the detailed implications are very difficult to derive explicitly, they have to complement the analysis of firm dynamics with some numerical results. Our paper contributes to this research programme by introducing a martingale approach to formulate the agent's incentive compatibility constraint. We then characterize the optimal contract through an ordinary differential equation. This characterization, unlike that using the discrete-time Bellman equation in Clementi and Hopenhayn(2006), allows for an analytic derivation of the impact of the model parameters on the optimal contract, firm dynamics, and well as investment sensitivity with respect to cash flows. While discrete-time models are adequate conceptually, a continuous-time setting may prove to be simpler and more convenient analytically.

Cooley and Quadrini(2001) present a theoretical model in which financial frictions explain several stylized facts about firm growth and generate FSD skewness. Desai, Gompers and Lerner(2003) suggest that financial constraints potentially induced by institutional factors, such as corruption and insufficient protection of property rights, have a negative impact on firms' entry and growth and generate FSD skewness, especially in developing countries.

A large body of literature, beginning with the distinguished work of Jovanovic(1982), rely on explain these industry dynamics as arising from learning about the technology or from persistent shocks to the technology. Examples of these models include Jovanovic(1982), Hopenhayn(1992), Hopenhayn and Rogerson(1993), Campbell(1998) and Campbell and Fisher(2000), among others. These models capture some of the empirical regularities mentioned above but they are unable to simultaneously account for both the size dependence (once we control for the age of the firm) and the age dependence (once we control for the size of the firm).

Our paper is part of a growing literature on dynamic optimal contracting models using recursive techniques that began with Green(1987), Spear and Srivastava(1987), Phelan and Townsend(1991), and Atkeson(1991) among others. As we mention above, this paper builds directly on the model of Clementi and Hopenhayn(2006), using the methodology developed by Sannikov(2008) and DeMarzo and Sannikov(2006). Other recent work that develops optimal dynamic agency models of the firm includes Albuquerque and Hopenhayn(2004), DeMarzo and Fishman(2004) and DeMarzo and Fishman(2007). However with the exception of Clementi and Hopenhayn(2006), these papers do not share our focus on financial constraint and firm dynamics.

Our model belongs to a large body of dynamic contract models of moral hazard, pioneering with the work of Radner(1985) and Rogerson(1985). Green(1987) and Spear and Srivastava(1987) develop the recursive representation the infinitely repeated principal-agent problem in discrete time. Spear and Srivastava(1987) introduce the agent's continuation value as a state variable, which contains the agent's behavior history, transforming the dynamic moral hazard into a standard dynamic programming problem. This methodology is taken by a large body of literatures, among them are Kocherlakota(1996), Hopenhayn and Nicolini(1997), Clementi and Hopenhayn(2006). These models suffer from complicated calculations, and in spite of its extremely conceptual intuition the detailed implications are very difficult to derive explicitly, they have to complement the analysis of firm dynamics with some numerical results. Sannikov(2008) develops a continuous-time analogue of the repeated principal-agent model, and provides a much cleaner characterization of the optimal contract through an ordinary differential equation. Our paper contributes to this body of literature by extend Sannikov(2008) methodology to debt contract and firm growth dynamics.

The remainder of this paper is organized as follows. The model is introduced in Section 2. In Section 3 we characterize the main properties of the optimal contract. Section 4 discusses the firm size dynamics. In section 5 we consider the investment sensitivity. And finally section 6 concludes.

2 The Model Setting

2.1 Technology and Preference

Time is continuous and infinite. There is one borrower (an entrepreneur) and one lender (a bank). The entrepreneur with initial net wealth A has a project which requires a fixed initial investment $I_0 > 0$ to set up a firm, and a continuous-time investment flow of working capital k_t at time t to run the project. Assume that $A \leq I_0$ and therefore the entrepreneur suffers from financial constraint. For simplicity and without loss of generality, we assume that $A = I_0 = 0$. The entrepreneur has enough initial wealth to invest in the fixed asset but have no working capital for the operation of the firm. To run the project, he requires a bank to finance the investment flow of working capital.

When the project is invested, it generates the continuous cash flow with mean k_t and volatility σk_t , or

$$dY_t = k_t dt + \sigma k_t dZ_t \tag{1}$$

where Z_t is standard Brownian motion. However, the cost function of the investment takes a quadratic form[1], i.e.,

$$c(k_t) = \frac{\theta k_t^2}{2}$$

where $\theta > 0$. It is worthy noting that the cash flow volatility increases with the amount of investment.(empirical reference). As we will see it below, the positive association between volatility and investment is the necessary condition to guarantee the working of financial constraint.

[1]In a previous version of the paper, we assmue the production technology takes a Cobb-Douglass form, while the cost function is linear, which follows Clementi and Hopenhayn(2006) more closely. Though analytically cubersome, our main results are derived under mild sufficient conditions.

Assume both the borrower and the lender are risk-neutral, but the borrower is less patient than the lender, that is, the borrower's discount factor, γ, is greater than the lender's discount factor, r.[2]

2.2 Agency Problem

We assume that the entrepreneur observes the cash flow Y_t , but the lender does not. In other words, such outcome is the private information for entrepreneur. Because of the information asymmetry, the entrepreneur can either conceal or divert the fund. We model the agency problem by allowing the agent to divert the cash flow for his own private benefit, as in DeMarzo and Fishman(2007) and DeMarzo and Sannikov(2006). Clementi and Hopenhayn(2006) considers an alternative source of moral hazard, that the outcome of the project is observable to the principal while the use of the fund is not. DeMarzo, Fishman, He and Wang(2008) and He(2008) consider shirking model, in which the shirking benefit is linear in the firm size. However, they share the same analytic properties.

Assume there exists agency cost and hence inefficiency when the entrepreneur diverts the firm's funds: the entrepreneur can only consume part of cash flows that he diverts from the firm. The agent receives a fraction $\lambda \in [0, 1]$ of the cash flows he diverts.

2.3 Long-term Contract

At time 0, a debt contract between the entrepreneur and the lender is specified as follow.

Definition 1 *A debt contract, (τ, I, k), specifies a termination time, τ , the cash flow right for the entrepreneur I_t , and the credit to entrepreneur and the investment level (firm size) k_t, that is based on the entrepreneur's report history $\{\hat{Y}_s; 0 \leq s \leq t\}$. Formally, I and k is a $\hat{Y}-$measurable continuous process, respectively, and τ is a $\hat{Y}-$measurable stopping time.*

We assume that the agent is essential to run the project. Once the project is terminated, the entrepreneur receives a payoff $Q \geq 0$ from the outside option, and the lender receives the expected liquidation payoff $L \geq 0$. Furthermore, We assume that at any time t the entrepreneur is liable for payment to the lender only to the extent of current revenue. That is, the firm is restricted at all time to a nonnegative cash flow or, $dI_t \geq 0$.

Both the entrepreneur and the lender are risk-neutral, take the same discount factor, and are able to commit to a long-term contract. The lender has unlimited capital.

Under the debt contract (τ, I, k), up to time $t \leq \tau$, the entrepreneur's instantaneous consumption equals

$$dC_t = \lambda[dY_t - d\hat{Y}_t] + dI_t, \quad dY_t - d\hat{Y}_t \geq 0 \tag{2}$$

[2] As DeMarzo and Sannikov(2006) and He(2008) make it clear, when $\gamma = r$, the principal postpone the agent's consumption "forever", and the optimal contract fails to exist.

When the project is terminated at time τ, the entrepreneur receives a payoff $Q \geq 0$ from an outside option. Hence the entrepreneur's total expected payoff from the contract at date 0 is given by

$$V_0(\hat{Y}) = E[\int_0^\tau e^{-\gamma t}(\lambda[dY_t - d\hat{Y}_t] + dI_t) + e^{-\gamma\tau}Q] \tag{3}$$

Similarly the lender's time-t cash flow equals

$$d\hat{Y}_t - dI_t - c(k_t)dt. \tag{4}$$

When the prefect is terminated, the lender receives the expected liquidation payoff $L \geq 0$. Hence the lender's total expected profit at date 0 is

$$D_0(\hat{Y}) = E[\int_0^\tau e^{-rt}(d\hat{Y}_t - dI_t - c(k_t)dt) + e^{-r\tau}L] \tag{5}$$

From the point and view of capital structure, V_0 is the equity value, while D_0 is the debt value.(further discussion)

3 Optimal Dynamic Contract

3.1 First-best Contract

When there exists no information asymmetry, and hence the lender observes the project's cash flow at any time t, the realized cash flow can be written into the contract. Since both the entrepreneur and the lender are risk neutral and share the same discount factor, there exists an optimal debt contract such that the sum of both agents' expected payoff is maximized or

$$k^* = argmax_k \ k - c(k) \tag{6}$$

The optimal investment satisfies the first-order condition

$$c'(k^*) = 1 \tag{7}$$

since $f(k_t)$ is concave. Hence we have

$$k^* = \frac{1}{\theta} \tag{8}$$

and the first-best instantaneous profit is $\frac{1}{2\theta}$.

3.2 Optimal Contract With Dynamic Moral Hazard

3.2.1 Borrower's Continuation Value and Incentive Compatibility

We solve the optimal dynamic contract problem following the methodology *a la* Sannikov(2008). In response to a debt contract (τ, I, k) , the entrepreneur chooses a feasible strategy that consists of his consumption choice and the income report in order to maximize his expected payoff. Below we formally define the feasible strategy of the entrepreneur.

Definition 2 *Given a contract* (τ, I, k) *, a feasible strategy is a pair of processes* $(C, \hat{Y})$ *adapted to* Y *such that*

1. $\hat{Y}_t$ *is continuous-time process, and* $Y_t - \hat{Y}_t$ *has bounded variation,*
2. C_t *is nondecreasing.*

The entrepreneur's strategy $(C, \hat{Y})$ is incentive compatible if it maximizes his lifetime expected utility in the class of all feasible strategies given a debt contract (τ, I, k) . As a result, we have the following definition.

Definition 3 *Given a debt contract* (τ, I, k) *, the entrepreneur's strategy* $(C, \hat{Y})$ *is incentive compatible if*

1. *the entrepreneur's strategy* $(C, \hat{Y})$ *is feasible;*
2. *the entrepreneur's strategy* $(C, \hat{Y})$

$$E[\int_0^\tau e^{-\gamma t} dC_t + e^{-\gamma \tau} Q_t | \digamma_t] \geq E[\int_0^\tau e^{-\gamma t} dC'_t + e^{-\gamma \tau} Q | \digamma_t] \tag{9}$$

for all the entrepreneur's feasible strategies $(C, \hat{Y}^{'})$ *, given an contract* (τ, I, k)*, where* $\digamma_t$ *is the* σ*-algebra of information associated with the process.*

Note that the entrepreneur's utility from the continuation of the project should be at least as large as the entrepreneur's outside option, Q, which he can receive at any time when the project is terminated. As the entrepreneur can always, for example, under-report and divert at rate rQ until a termination time, any incentive compatible strategy would yield the entrepreneur utility of at least Q.

(Clementi and Hopenhayn(2006), this constraint may bind in a discrete-time setting because of a limit to the amount the agent can steal per period.??)

A debt contract is incentive compatible if the entrepreneur's strategy $(C, \hat{Y})$ is incentive compatible given the contract (τ, I, k). Denote the incentive compatible contract as $(\tau, I, k, C, \hat{Y}, Y)$. The optimal contracting problem is to find an incentive compatible contract $(\tau, I, k, C, \hat{Y}, Y)$ that maximizes the lender's profit subject to delivering the entrepreneur an required flow of payoff rQ.

Now we show that it is sufficient to look for an optimal contract in which the agent chooses to report cash flows truthfully. To characterize the optimal contract recursively, we define the borrower's continuation value at time t if he tells the truth.

Definition 4 *Given the history of reports at time t,* $\{\hat{Y}_s; s \leq t\}$*. The entrepreneur's continuation value* W_t *is defined as the total expected discounted value that he receives at time t, from transfers and termination, if the entrepreneur takes the truth-telling strategy after time t:*

$$W_t(\hat{Y}) = E[\int_t^\tau e^{-\gamma(s-t)} dI_s + e^{-\gamma(\tau - t)} Q | \digamma_t] \tag{10}$$

Definition 5 *Define V_t to be the lender's time-0 total expected value conditional on the his information at time t, from transfers and termination utility, if he always tells the truth:*

$$V_t = E[\int_0^\tau e^{-\gamma s} dI_s + e^{-\gamma\tau} Q | \mathcal{F}_t] \tag{11}$$

where $\mathcal{F}_t$ is the information available to the entrepreneur at time t.

It is easy to see the process V_t is a square-integrable $\mathcal{F}_t$-martingale and can be written as

$$\begin{aligned} V_t &= \int_0^t e^{-\gamma s} dI_s + e^{-\gamma t} E[\int_t^\tau e^{-\gamma(s-t)} dI_s + e^{-\gamma(\tau - t)} Q | \mathcal{F}_t] \\ &= \int_0^t e^{-\gamma s} dI_s + e^{-\gamma t} W_t(Y) \end{aligned} \tag{12}$$

Using the famous Martingale Representation Theorem and Ito Lemma, we can written the process W_t as an $\mathcal{F}_t$-measurable diffusion process.

Lemma 1 *There exists an $\mathcal{F}_t$-predictable process β_t, $0 \leq t < \tau$, such that*

$$dW_t(\hat{Y}) = \gamma W_t dt - dI_t + \beta_t(\hat{Y}_t)(d\hat{Y}_t - k_t dt) \tag{13}$$

Proof. See Appendix. ■

The process $\beta_t(\hat{Y}_t)$ in formula (13) is the sensitivity of the continuation value with respect to the report. That is, the marginal continuation value of the entrepreneur's report. Note that λ is the entrepreneur's marginal cost of report. The entrepreneur has incentives not to divert cash flows if he gets at least each reported dollar, that is, if $\beta_t \geq \lambda$. If this condition holds for all t, then the entrepreneur's payoff will always integrate to less than his continuation value if he deviates. If this condition fails on a set of positive measure, the entrepreneur can obtain at least a little bit more than his continuation value if he under-reports cash when $\beta_t < \lambda$. We summarize the conclusion as the following proposition.

Proposition 1 *The truth-telling strategy is incentive compatible if and only if $\beta_t \geq \lambda$ for all $t \leq T$.*

Proof. see appendix. ■

Since the entrepreneur can not save, he tells the truth whenever $\beta_t \geq \lambda$ for all $t \leq T$. The linearity of the incentive comparability condition simplifies the dynamic programming approach to determine the optimal contract for the principal to deliver the agent any value W . Here we discuss informally. The proof of Proposition formalizes our discussion below.

3.2.2 Derive the Optimal Contract

Given the entrepreneur's continuation value W and the truth-telling strategy. Let $D(W)$ be the lender's highest expected payoff that can be obtained from an incentive compatible debt contract, providing the entrepreneur with utility equal to a W .

We observe that transferring lump-sum dI from the lender to the entrepreneur with continuation value W, moves a contract to that of the entrepreneur's continuation value $W - dI$. The efficiency implies that

$$D(W) \geq D(W - dI) - dI, \tag{14}$$

which shows that the marginal cost of delivering the entrepreneur his continuation value can never exceed the cost of an immediate transfer in terms of the lender's utility. That is

$$D'(W) \geq -1 \tag{15}$$

Define W^* as the lowest value W such that $D'(W) = -1$. That is, $W^* = inf\ \{W : D'(W) = -1\}$. As we prove in the Appendix, $D(W)$ is concave, hence it is optimal to pay the borrower as follows.

Proposition 2 *When $W \leq W^*$, $dI_t = 0$; and when $W > W^*$, $dI_t = W - W^*$*

It is easy to see Proposition 2 intuitively. When the entrepreneur's continuation value W is small, the agency problem becomes serious, and the lender prefers to provide incentive by delaying the transfer to the entrepreneur. When the entrepreneur's continuation value W is large, the entrepreneur takes a large proportion of the firm's cash flow rights, and the agency problem becomes relatively mild. Hence the incentive of exercising transfer dominates that of delaying.

It is worthy noting that the transfer policy is quite similar with that of the standard short-term debt. The standard short-term debt contract specifies a contingent transfer policy that the entrepreneur gains the residual claim right when the project's cash flow is not less that the pre-specified repayment, and otherwise the lender gains the whole cash flow rights. The long-term contract specifies a similar contingent transfer policy that the entrepreneur gains the residual claim right $W - W^*$ when his continuation value is not less than a threshold W^*. Otherwise, the lender takes all cash flow rights.

When $W \in [Q, W^*]$, the entrepreneur consumes nothing, and if he has some luck to perform well in the future, his performance is valued in the continuation utility. If he is lucky enough that the project gets continually positive shocks and the realizations of dZ_t are continually positive, and hence his continuation value $W \geq W^*$, he will gain consumption. If he suffers from bad luck that the project gets continually negative shocks and the realizations of dZ_t are continually negative, and hence his continuation value $W < Q$, the lender will never invest the project, and the project is terminated. When $W \in [Q, W^*]$ and the entrepreneur tells the truth, his continuation utility evolves according to

$$dW_t = \gamma W_t dt + \beta_t \sigma k_t dZ_t. \tag{16}$$

In this region, we need to characterize the optimal choice of process $\beta(t)$, which determines the sensitivity of the entrepreneur's continuation value with respect to his report. Because at the optimum the lender should earn an instantaneous total return equal to the discount rate, r, we have the following Hamilton highest total expected payoff:

$$D(W) = \max_{\beta_t \geq \lambda,\ k_t} E[dY_t - c(k_t)dt + \frac{D(W(t+dt))}{1+rdt}|F_t] \tag{17}$$

Using Ito lemma, equation (17) becomes

$$rD(W) = \max_{\beta \geq \lambda,\ k} [k - \frac{\theta k^2}{2} + \gamma W D'(W) + \frac{D''(W)}{2}\beta^2\sigma^2 k^2] \tag{18}$$

We prove in the Appendix that $D(W)$ is strictly concave in the region $[Q, W^*]$, setting $\beta = \lambda$ for all $t \leq \tau$ is optimal.

The concavity of the objective function in k in the RHS of the Bellman equation also implies that the optimal choice of k is given as a solution to

$$\hat{k} = \frac{1}{\theta - \lambda^2\sigma^2 D''(W)} \tag{19}$$

Substituting the first-order condition into the above HJB function, we have the following ODE

$$2(\theta - \lambda^2\sigma^2 D''(W))(rD(W) - \gamma W D'(W)) = 1, W \in [Q, W^*] \tag{20}$$

Given the strictly concavity of $D(W)$, $1/(\theta - D''(W)\lambda^2\sigma^2) < 1/\theta$. Hence

$$\hat{k} < k^* \tag{21}$$

That is, when the entrepreneur's continuation value is less than W^* , the project is underinvested, and hence the entrepreneur suffers the financial constraint.

When $W > W^*$, $D(W) = D(W^*) + W^* - W$, hence $D''(W) = 0$. Meanwhile, the super contact condition requires the second derivatives match at the boundary, that is, $D''(W^*) = 0$. Substituting it into equation (19) , we have

$$\hat{k} = k^*. \tag{22}$$

When the entrepreneur's continuation value is no less than W^*, the project is free from financial constraint, and the investment attains the first-best level. We summerize the conclusion as the following proposition.

Proposition 3 *The optimal contract takes the following form:* W_t *evolves according to*

$$dW_t = \gamma W_t dt - dI_t + \lambda(dY_t - k_t dt) \tag{23}$$

When $W_t \in [Q, W^*]$*, the investment is less than the socially optimal level, and* $dI_t = 0$*; When* $W \geq W^*$*, the investment attains the socially optimal level, and* $dI_t = W - W^*$*. The principal's expected payoff is given by a concave function* $D(W)$*, satisfying (20) on the inteval* $[Q, W^*]$*,* $D'(W) = -1$ *for* $W \geq W^*$ *and boundary conditions* $D'(W^*) = -1$ *and* $D''(W^*) = 0$*.*

[Insert Figure 1 here]

From equation (19) the investment inefficiency relates to λ^2, σ^2, and $D''(W)$. Consider the entrepreneur's diversion efficiency λ. It measures the quality of outside investment protection. A large λ means the weak legislature and legal enforcement of the outside investor's protection, such that the entrepreneur easily divert the project's cash flow as his private benefit. Because of information asymmetry, the truth-telling mechanism works costly such that $\beta \geq \lambda$. If there exists no agency problem ($\lambda = 0$), the project investment attains the socially optimal level by letting $\beta = \lambda = 0$ and the second RHS term in equation (19) equals zero.

When the legal protection of outside investor is poor, the entrepreneur easily divert and conceal the project's cash flow as his private benefit. To protect her benefit, the lender decreases the investment to the project that performs relatively poor, even though the project should have been invested at the socially optimal level. Hence λ is also a measure of degree of financial constraint.

The concavity of plays a critical role in deriving Proposition 3. It is intuitive to see this in the views of either riskness or incentiveness. From the point and view of riskness, the concavity of $D(W)$ in $[Q, W^*]$ means the lender is induced as if she is relatively *risk-averse*, and hence invests less with respect to that when she is *risk-neutral*.

Consider the intuition in the view of incentiveness. When the project performs continually well, as a bonus the entrepreneur holds increasingly more the cash flow right in the future (futures or options??)(note that the entrepreneur still holds zero of the current cash flow right and consumes nothing), such that the entrepreneur's continuation value W increases continually, and hence mitigates the agency problem, which results in the less cash flow rights to the lender in the future and hence the concavity of $D(W)$ in $[Q, W^*]$. The continual good performance of the project mitigates the agency problem and increases the investment. When the entrepreneur's continuation value is large enough, the financial constraint does not work, and the investment attains the socially optimal level.

The entrepreneur prefers to investing more because of limited liability. The lender can use it as a incentive mechanism. When the project performs continually well, as a bonus, the lender increases the investment; and when the project performs continually poor, as a punishment, the lender decreases the investment and extremely stops the project.

4 Firm Size Dynamics

This section considers some of the implications of the model for firm dynamics. Following Clementi and Hopenhayn(2006), define the working capital k_t as the firm size.

4.1 Capital Advancement Policy

Consider the relationship between the firm size k_t and the entrepreneur's continuation value W_t. Solving the derivative on both sides of equation (18) with respect to W_t and using the envelope theorem, we find

$$(r-\gamma)D'(W_t) - \gamma D''(W_t)W_t = \frac{D'''(W_t)}{2}\lambda^2\sigma^2 k_t^2 \tag{24}$$

Combining the first-order condition (19) with ODE (20), we get the policy function:

$$k_t = F(W_t) = 2(rD(W_t) - \gamma W_t D'(W_t)) = \frac{1}{\theta - \lambda^2\sigma^2 D''(W_t)} \tag{25}$$

Then,

$$\frac{\partial k_t}{\partial W_t} = F'(W_t) = 2(r-\gamma)D'(W_t) - 2\gamma D''(W_t)W_t \tag{26}$$

$D(W_t)$ is not necessarily decreasing on all its domain, as the optimal contract in general is not renegotiation-proof. Hence (26) implies that when W_t is small, k_t may decrease with W_t. In particular, when $-\gamma D''(W_t)W_t < (\gamma - r)D'(W_t)$, $F'(W_t) < 0$. However, in the "renegotiation-proof" region, i.e., $D'(W_t) \leq 0$, we have $D'''(W_t) > 0$, thus $F'(W_t) > 0$. As $D'(W_t)$ decreases with W_t, if $D'(W_1) \leq 0$, then for any $W > W_1$, $F'(W) > 0$.

Definition 6 $\widehat{W}$ *is the starting point of the "renegotiation-proof" region, i.e.,* $D'(\widehat{W}) = 0$.

Proposition 4 *If* $-\gamma D''(Q)Q < (\gamma - r)D'(Q)$*, then there exists some* $\widetilde{W}$*,* k_t *decreases with* W_t *when* $W_t \leq \widetilde{W}$*; If* $W_t > \widehat{W}$*,* k_t *increases with* W_t*.*

This is consistent with Clementi and Hopenhayn(2006)'s result. But the intuition may be different.

We also infer from (26) when r equals γ, the downside effect is gone and the decreasing region vanishes. But again, in that case, the optimal contract may not exist. However, when r is close to γ, numerical example shows that k_t increases with W_t.

[Insert Figure 2 here]

4.2 Size Dependence

The empirical regularities of firm dynamics show that, conditional on age, the dynamics of firms (growth, volatility of growth, job creation, job destruction and exit) are negatively related to the size of firms. [3] These results agree with empirical studies of Evans(1987) and Hall(1987), among others.

Combining the first-order condition (19) with ODE (20), we get the policy function:

$$k_t = F(W_t) = 2(rD(W_t) - \gamma W_t D'(W_t)) = \frac{1}{\theta - \lambda^2\sigma^2 D''(W_t)} \tag{27}$$

[3] However, in our model, W_t is the only dimension of heterogeneity, thus we can only generate *unconditional* dependence of firm dynamics on size and age.

Using Ito's lemma, we find the diffusion function of k_t:

$$\begin{aligned} dk_t &= F'(W_t)dW_t + \frac{F''(W_t)}{2}(dW_t)^2 \\ &= (\gamma W_t F'(W_t) + \frac{F''(W_t)}{2}\lambda^2\sigma^2 k_t^2)dt + \lambda\sigma k_t F'(W_t)dZ_t \\ &= (r-2\gamma)\lambda^2\sigma^2 k_t^2 D''(W_t)dt + 2\lambda\sigma k_t((r-\gamma)D'(W_t) - \gamma W_t D''(W_t))dZ_t \end{aligned} \tag{28}$$

where the third equality comes from equation (24). Hence,

$$\begin{aligned} \frac{dk_t}{k_t} &= (r-2\gamma)\lambda^2\sigma^2 k_t D''(W_t)dt + 2\lambda\sigma((r-\gamma)D'(W_t) - \gamma W_t D''(W_t))dZ_t \\ &= (r-2\gamma)(\theta k_t - 1)dt + 2\lambda\sigma((r-\gamma)D'(W_t) - \gamma W_t D''(W_t))dZ_t \\ &= \mu(k_t, W_t)dt + \delta(k_t, W_t)dZ_t \end{aligned} \tag{29}$$

where the second equality comes from the first-order condition (19), and $\mu(k_t, W_t)$ is the expectation of firm growth rate, and $\delta(k_t, W_t)$ is the volatility of growth rate.

It is easy to show that

$$\frac{\partial \mu(k, W)}{\partial k} = (r-2\gamma)\theta < 0 \tag{30}$$

Proposition 5 *The growth rate decreases with the firm size.*

However, it's quite difficult to sign $\partial\delta(k, W)/\partial k$.

$$\begin{aligned} \frac{\partial \delta(k, W)}{\partial k} &= 2\lambda\sigma((r-2\gamma)D''(W) - \gamma W D'''(W))\frac{\partial W}{\partial k} \\ &= 2\lambda\sigma((r-2\gamma)D''(W) - \gamma W D'''(W))/F'(W) \end{aligned} \tag{31}$$

When W is small, it might be the case that $F'(W) < 0$. Then we know from (25) $D'''(W) < 0$, hence $\partial\delta(k, W)/\partial k < 0$.On the other side, when W is close to W^*, we also get $\partial\delta(k, W)/\partial k < 0$. In the limit $W = W^*$, we have $D''(W^*) = 0$ and $D'(W^*) = -1$. From (25) and (26), $D'''(W^*) > 0$ and $F'(W^*) > 0$, hence $\partial\delta(k, W^*)/\partial k < 0$.However, over the region $[Q, W^*]$, we are not able to prove explicitly that the volatility of growth decreases with the firm size. Therefore, we complement our analysis with numerical exercise.

[Insert Figure 3 here]

Finally, in the continuous-time model, public randomization is unnecessary–when $W_\tau = Q$, the project is liquidated. Hence, the conditional probability of survival $\Pr(\tau > \widetilde{t}|W_t)$ is increasing in the agent's continuation payoff W_t. Therefore

Proposition 6 *When $W_t > \widehat{W}$, the conditional hazard rate for exit,* $\Pr(\tau > \widetilde{t}|k_t)$*, decreases with firm size k_t.*

However, different from the discrete-time model, even if W reaches W^*, the possibility of liquidation, though small, always exists, as the potential loss is unbounded in the continuous-time model.

4.3 Age Dependence

Consider the implications of firm age for firm dynamics. Empirical results also shows, conditional on size, the dynamics of firms (growth, volatility of growth, job creation, job destruction and exit) are negatively related to the age of firms.

When $D'(W^*) = -1 \in [Q, W^*]$, we know from (28)

$$E(dk_t) = (r - 2\gamma)\lambda^2\sigma^2 k_t^2 D''(W_t)dt > 0$$

Hence, firm size increases on average as the firm ages. Then it is obvious that similar results hold for firm's age, on average. Formal derivation is difficult, but numerical results are in line with the empirical regularities.

Following Clementi and Hopenhayn(2006), we use a finite, large number of firms in the numerical exercise, assuming all firms start at the same initial value W_0, which is close to agent's outside option Q. Time increment is discrete and small, the step $h = 0.01$.At each time, when the firms' shocks are revealed, W_t evolves according to (23) and k_t according to (28). Then, exit hazard rate, average size, average growth rate and volatility of growth are obtained.[4]

[Insert Figure 4 here]

As mentioned above, the hazard rate may not necessarily converge to zero, as the potential loss is unbounded here.

4.4 Evolution of Firm Size Distribution

Conventional wisdom confirms that the firm size distribution is stable and approximately lognormal. However, Cabral and Mata(2003) document two stylized facts about the FSD, using a data set of Portuguese manufacturing firms: the distribution of young firms is very skewed to the right, that is, most of the mass is on small firms; and the skewness tends to diminish monotonically with firm age, that is, the distribution of older firms is more symmetric than that of young firms. Then they use a simple two-period competitive model, to demonstrate that it is financial constraint, rather than selection, could well account for the evolution.

However, the theoretical model of Cabral and Mata(2003) is a little too simple and stylized. Technically speaking, there are no optimization problems for the firms to solve, and more importantly, financial constraints are exogenous. Therefore, it is necessary to introduce a more sophisticated dynamic model. Fortunately, our complex model is able to replicate the evolution of firm size distribution that is observed in Cabral and Mata(2003).

[4] The time series are averaged in the cross section, among the surviving firms, and then averaged over 50 steps, i.e. $T = 0.5$.

[Insert Figure 5 here]

We use a sample of 200000 firms, starting with the same initial value (for the borrower), W_0,[5], and simulate the evolution of their size according to the optimal debt contract.

5 Comparative Static Analysis

6 Summary and Conclusion

7 Appendix

7.1 Entrepreneur's Incentive Compatibility Condition

Proof of Lemma 1. $W_t(\hat{Y})$ is determined by the entrepreneur's reported cash flow $\{\hat{Y}_s; s \leq t\}$. One possibility is that $\{\hat{Y}_s; s \leq t\}$ equals the realized cash flow. That is, the entrepreneur tells the truth. In this case, we have

$$V_t = \int_0^t e^{-\gamma s} dI_s + e^{-\gamma t} W_t(\hat{Y}) \tag{32}$$

It can be written as a stochastic differential equation

$$dV_t = e^{-\gamma t} dI_s + e^{-\gamma t} dW_t(\hat{Y}) - \gamma e^{-\gamma t} W_t(\hat{Y}) dt \tag{33}$$

V_t is a martingale, hence by the martingale representation theorem, there exists a $\beta_t(\hat{Y}_t) > 0$ such that

$$dV_t = e^{-\gamma t} \beta_t(\hat{Y}_t)(d\hat{Y}_t - k_t)dt \tag{34}$$

We complete proof by combining equation (33) and (34). ■

Before the proof of Proposition 1, we define an auxiliary gain process for the borrower

Definition 7 *When the entrepreneur reports cash flows equal $\{\hat{Y}_s; s \leq t\}$ before time t, and from then on she tells the truth, her lifetime total revenue expected at time t is*

$$\hat{V}_t = \int_0^t e^{-\gamma s} dI_s(\hat{Y}_s) + \lambda[dY_t - d\hat{Y}_t]) + e^{-\gamma t} W_t(\hat{Y}) \tag{35}$$

Proof of Proposition 1. Writing equation (35) as a stochastic differential equation, using Ito Lemma, and combining equation (13), we have ■

$$\begin{aligned} d\hat{V}_t &= e^{-\gamma t}((dI_s(\hat{Y}_s) + \lambda[dY_t - d\hat{Y}_t]) + dW_t(\hat{Y}) - \gamma W_t(\hat{Y})dt) \\ &= e^{-\gamma t}((dI_s(\hat{Y}_s) + \lambda[dY_t - d\hat{Y}_t]) - \gamma W_t(\hat{Y})dt + \gamma W_t(\hat{Y})dt - dI_s(\hat{Y}_s) + \beta_t(\hat{Y}_t)(d\hat{Y}_t - k_t dt)) \\ &= -e^{-\gamma t}((\beta_t(\hat{Y}_t) - \lambda)(dY_t - d\hat{Y}_t) + \beta_t(\hat{Y}_t)\sigma k_t dZ_t) \end{aligned} \tag{36}$$

[5] We have also started with a uniform distribution of the value, and the result is similar.

As we have assumed that the entrepreneur has only the incentive to divert firm's cash flows, $dY_t - d\hat{Y}_t \geq 0$ supermartingale if $\beta_t \geq \lambda$. Hence we have

$$V_0(Y) = \hat{V}_0 \geq E[\hat{V}_\infty | F_0] = V_0(\hat{Y}) \tag{37}$$

The expected revenue if she tells the truth is not less than that if she takes any strategy other than truth-telling. That is the truth-telling strategy is incentive compatible. Otherwise, $\hat{V}_t$ is a submartingale if $\beta_t < \lambda$. Then there exists a time t such that

$$E[\hat{V}_t | F_0] > \hat{V}_0 = V_0(Y) \tag{38}$$

Even if there exists such a strategy that the entrepreneur reports $\{\hat{Y}_s : s \leq t\}$ before t and tells the truth then after, the expected revenue if she take such a strategy is greater than that if she takes the truth-telling strategy at any time. In this case, the truth-telling strategy is incentive incompatible.

7.2 Concavity of the Optimal Contract

Evaluating (24) at the upper-boundary W^*, and using $D'(W^*) = -1$, $D''(W^*) = 0$, and $k = k^*$, we get

$$\frac{D'''(W)}{2} \lambda^2 \sigma^2 k^{*2} = \gamma - r > 0$$

therefore there exists some ϵ, for $W \in (W^* - \epsilon, W^*)$, $D''(W) < 0$.

Suppose that there exists some $\widetilde{W} \leq W^* - \epsilon$ such that $D''(\widetilde{W}) = 0$. Without loss of generality, assume that $\widetilde{W} = \sup\{W \in [Q, W^*] : D''(\widetilde{W}) = 0\}$. Thus, $D''(\widetilde{W} + \epsilon/2) < 0$ and $D'''(\widetilde{W}) < 0$. Then we know from equation (24) that $D'(\widetilde{W}) > 0$.

Evaluating equation (20) at $\widetilde{W}$, we have

$$2(\theta - \lambda^2 \sigma^2 D''(\widetilde{W}))(rD(\widetilde{W}) - \gamma W D'(\widetilde{W})) = 2\theta(rD(\widetilde{W}) - \gamma W D'(\widetilde{W})) = 1$$

hence

$$rD(\widetilde{W}) = \frac{1}{2\theta} + \gamma W D'(\widetilde{W}) > \frac{1}{2\theta}$$

which is impossible because $\frac{1}{2\theta}$ is the first best instantaneous profit. Therefore $D(W)$ is strictly concave over $[Q, W^*]$.

7.3 Verification of the Optimal Contract

Consider any incentive-compatible contract. For any $t \leq \tau$, define the auxiliary gain process as

$$G_t = \int_0^t e^{-rs}(dY_s - dI_s - c(k_s)ds) + e^{-rt} D(W_t)$$

Under incentive-compatible condition, W_t evolves according to (13). Then, from Ito's lemma,

$$\begin{aligned} e^{rs}dG_t &= (k_t - c(k_t) + \gamma W D'(W_t) + \frac{D''(W_t)}{2}\beta^2\sigma^2 k_t^2)dt - \\ &\quad (1 + D'(W_t))dI_t + (1 + \beta D'(W_t))\sigma k_t dZ_t \end{aligned}$$

We will prove that the drift of G_t is non-positive. For the first piece, the first-order condition (19) gives the optimal investment policy; because of the concavity of $D(W)$, $\beta = \lambda$ is optimal under incentive-compatible condition. Then from (18), the first piece is 0, while other incentive-compatible contracts will make it nonpositive.

For the second piece, recall that $D'(W) \geq -1$, hence $-(1 + D'(W_t))dI_t \leq 0$. However, under the optimal contract, $dI_t = 0$ when $D'(W_t) > -1$, and $dI_t > 0$ when $D'(W_t) = -1$, thus $(1+D'(W_t))dI_t = 0$.

Therefore, under any incentive-compatible contract, the drift of G_t is non-positive, and is zero under the optimal contract. It's obvious that $(1+\beta D'(W_t))\sigma k_t$ is bounded, thus G_t is a supermartingale, and a martingale under the optimal contract until time τ.

Then, for all $t < \infty$, the principal's payoff for an arbitrary incentive compatible contract is

$$\begin{aligned} & E[G_\tau] \\ = \; & E[G_{t\wedge\tau} + \mathbf{1}_{t\leq\tau}(\int_t^\tau e^{-rs}(dY_s - dI_s - c(k_s)ds) + e^{-r\tau}L - e^{-rt}D(W_t))] \\ = \; & E[G_{t\wedge\tau}] + e^{-rt}E[\mathbf{1}_{t\leq\tau}(E_t[\int_t^\tau e^{-r(s-t)}(dY_s - dI_s - c(k_s)ds) + e^{-r(\tau-t)}L] - D(W_t))] \\ \leq \; & G_0 + e^{-rt}\frac{1}{2r\theta} \end{aligned}$$

where the first term of the inequality comes from the supermartingale property of $G_{t\wedge\tau}$, and the second term comes from the fact that $\frac{1}{2r\theta}$ is the first-best profit of the principal if he owns the project solely without agency problem. Therefore, letting $t \to \infty$, $E[G_\tau] \leq G_0$.

Finally, $E[G_\tau] = G_0$ under the optimal contract as then G_t is a martingale until time τ.

7.4 Comparative Static Analysis

The generalized Feynman-Kac formula, i.e., Lemma 4 in DeMarzo and Sannikov(2006), holds in the present setting.

Lemma 2 *Suppose W_t evolves according to*

$$dW_t = \gamma W_t dt - dI_t + \lambda\sigma k_t dZ \tag{39}$$

until time τ, when W_t reaches Q, where I_t is a nondreasing process that reflects W_t at W^*. Let b be a real number and $g : [Q, W^*] \to \mathbb{R}$ be a bounded function. Then the same function $G : [Q, W^*] \to \mathbb{R}$ both solves

$$rG(W) = g(W) + \gamma W G'(W) + \frac{1}{2}\lambda^2\sigma^2 k^2 G''(W)$$

with boundary conditions $G(Q) = L$ *and* $G'(W^*) = -b$, *and satisfies*

$$G(W_0) = E[\int_0^\tau e^{-rt} g(W_t) dt - b \int_0^\tau e^{-\gamma t} dI_t + e^{-r\tau} L]$$

The proof is just similar to DeMarzo and Sannikov(2006).

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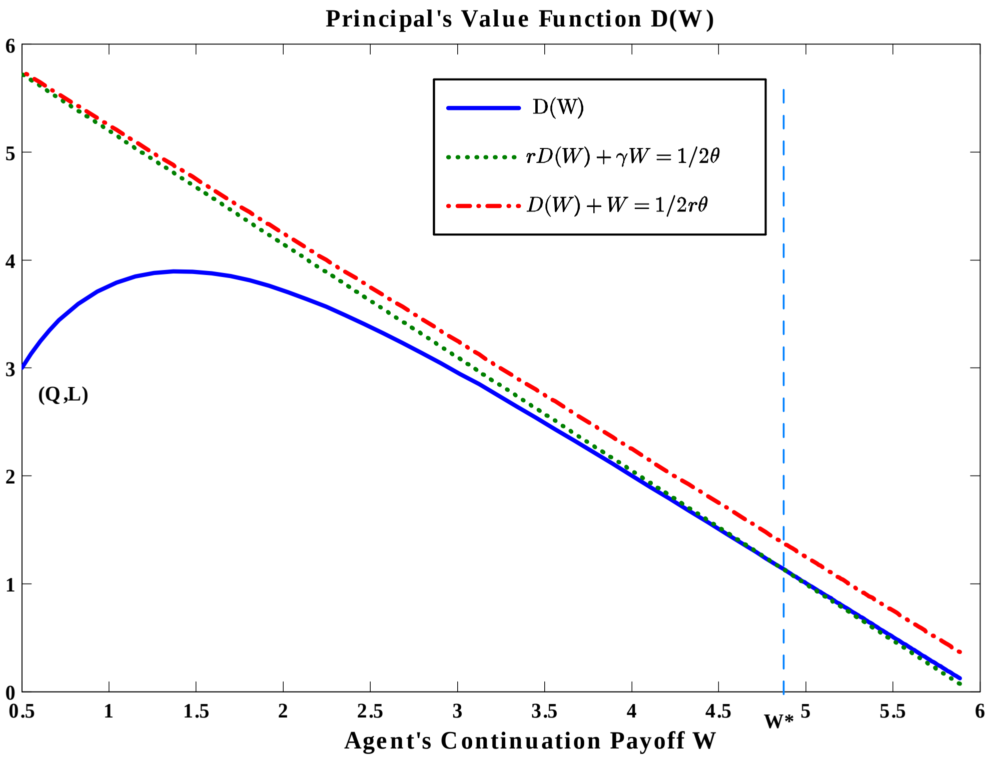


Figure 1: **Principal's Value Function** $D(W)$.The solid line is principal's value function $D(W)$.The dotted line is $rD(W)+\gamma W=1/2\theta$.The dash-dotted line is the first-best line, $D(W)+W=1/2r\theta$.The parameters are $r=0.04$, $\gamma=0.042$, $\theta=2$, $\lambda=0.8$, $\sigma=1.2$, $Q=0.5$ and $L=3$. The upper endogenous boundary $W^*=4.88$.

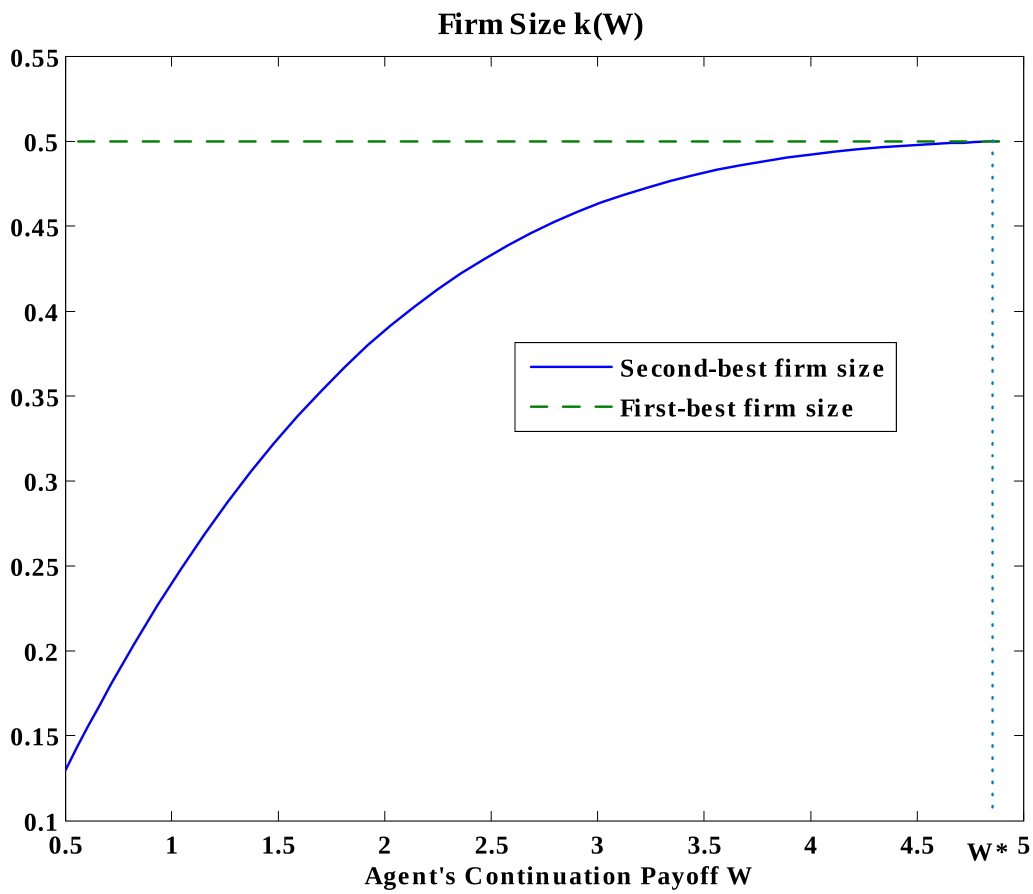


Figure 2: **Firm Size** $k(W)$.The solid line is the second-best firm size $k(W)$.The dashed line is the first-best firm size, $1/\theta$. The parameters are $r = 0.04$, $\gamma = 0.042$, $\theta = 2$, $\lambda = 0.8$, $\sigma = 1.2$, $Q = 0.5$ and $L = 3$. The upper endogenous boundary $W^* = 4.88$.

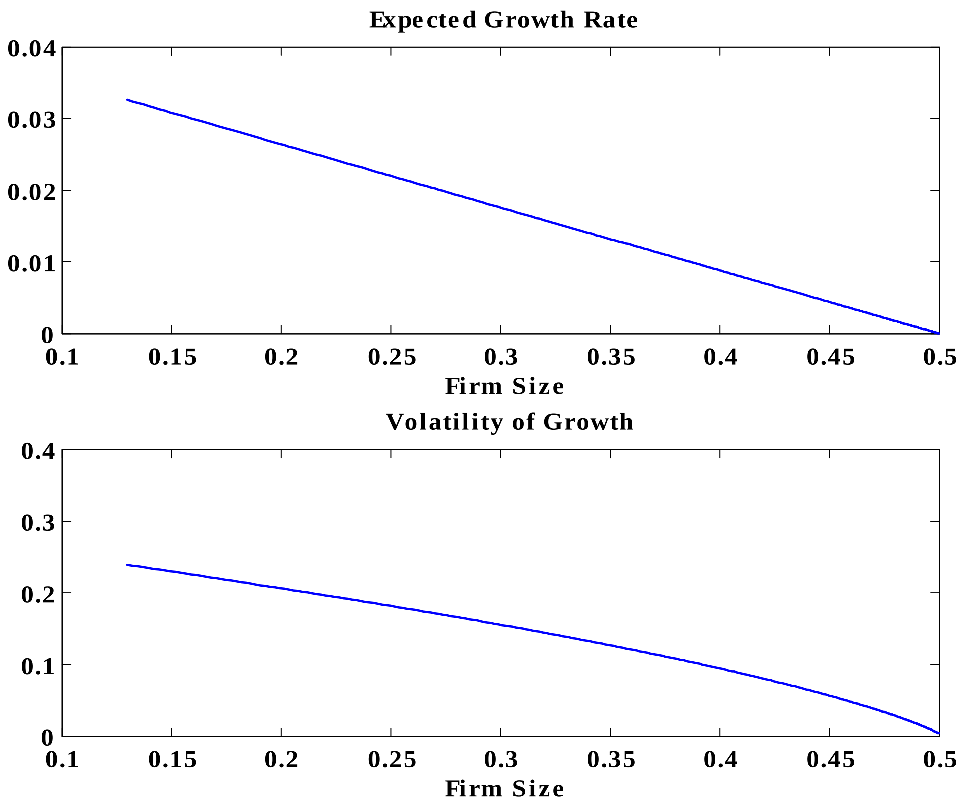


Figure 3: **Expected Growth Rate and Volatility of Growth Rate.** The top panel shows the linear and decreasing relationship between the expected growth rate, $\mu(k, W)$, and the firm size, k. The bottom panel shows the monotonically decreasing relationship between the volatility of growth rate, $\delta(k, W)$, and the firm size, k. The parameters are $r = 0.04$, $\gamma = 0.042$, $\theta = 2$, $\lambda = 0.8$, $\sigma = 1.2$, $Q = 0.5$ and $L = 3$.

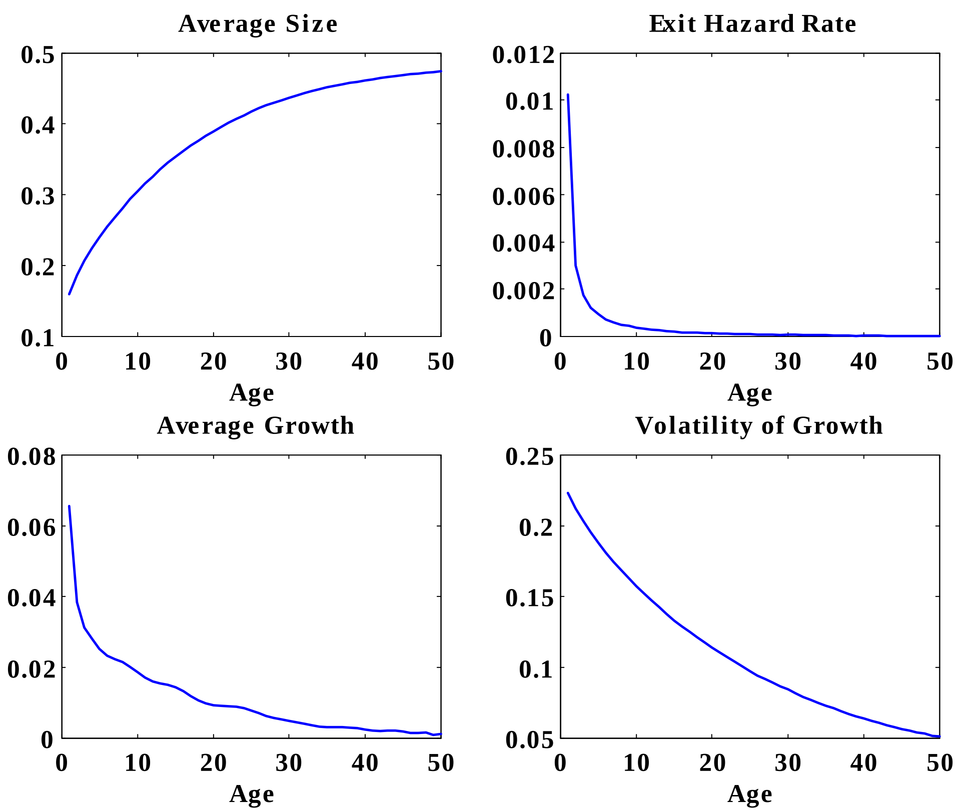
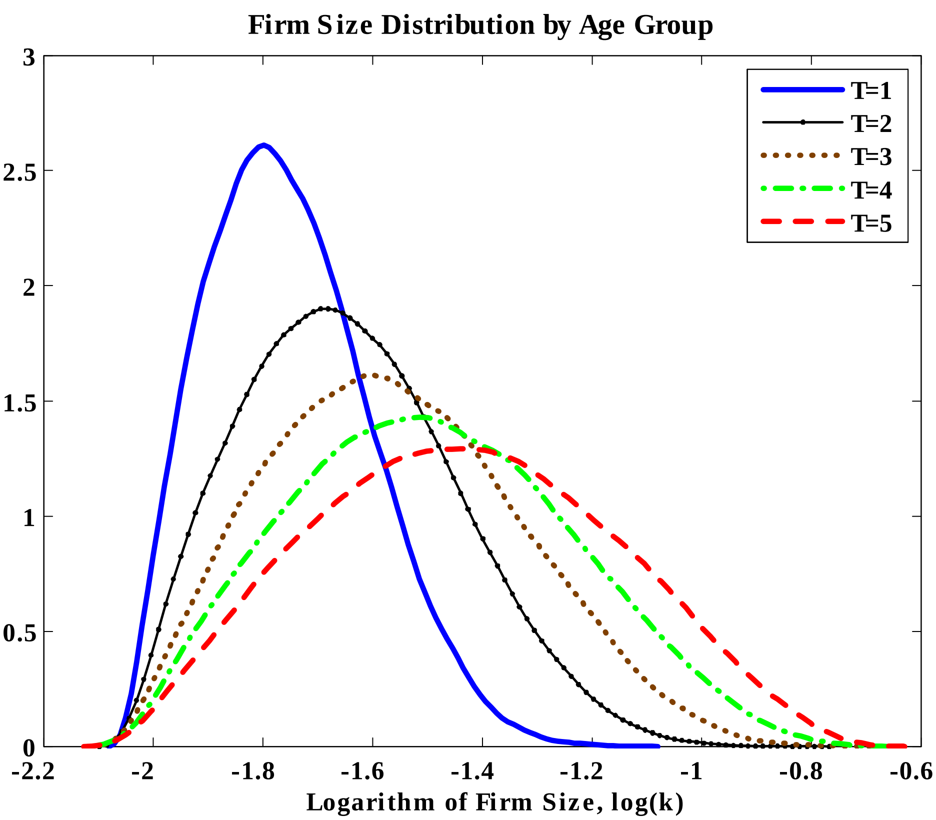


Figure 4: **Age Dependence.**


Firm Size Distribution by Age Group
3
2.5
2
1.5
1
0.5
0
-2.2
-2
-1.8
-1.6
-1.4
-1.2
-1
-0.8
-0.6
Logarithm of Firm Size, log(k)
T=1
T=2
T=3
T=4
T=5